# A HELLY TYPE THEOREM FOR *d*-STARSHAPED SETS

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Se demonstrează o teoremă de tip Helly despre intersecția mulțimilor compacte d-stelate în spațiul Minkowski $R^n$ cu metrica $d(x,y)=\|x-y\|$. Din această teoremă rezultă renumita teoremă Helly despre intersecția mulțimilor convexe [1] şi o variantă a teoremei Helly despre intersecția mulțimilor stelate în spațiul euclidian $E^n$ demonstrată în [2] .

We denote by $R^n$ the *n*-dimensional real linear space with the metric $d(x,y)=\|x-y\|$.

Recall that the set $S\subset R^n$ is said *starshaped* (*d-starshaped* [3]) if there exists a point $x\in S$ such that for each point $y\in S$ linear segment (*d-segment*)

$$[x,y]=\{z\in R^n \mid z=(1-\lambda)x+\lambda y,\ 0\le\lambda\le 1\}$$

$$\left(\langle x,y\rangle=\{z\in R^n \mid d(x,z)+d(z,y)=d(x,y)\}\right)$$

lies in *S*. The set of all such points $x\in S$ is the kernel (*d*-kernel) of *S* and is denoted by *kern S* (*d-kern S*). A *d*-starshaped set *S* is also starshaped and *d-kern S* ⊂ *kern S*.

In what follows, if $x\in S$, let *S*(*x*) denote the set of all points $y\in S$ such that $\langle x,y\rangle\subset S$. If *S* is *d*-starshaped then clearly $\bigcap_{x\in S} S(x)=d-kern\ S$.

A set $S\subset R^n$ is said to be *d-convex* if $\langle x,y\rangle\subset S$ for any points $x,y\in S$. It is true that *S* = *d-kern S* if and only if the set $S\subset R^n$ is d-convex. Let's denote the intersection of all *d*-convex sets of $R^n$ containing *S* with *d-conv S* which is called *d-convex hull* of the set *S* ([4]).

Let $F=\{S_i\}_{i\in I}$ be a arbitrary fixed family of *d*-starshaped compact sets in $R^n$ and let $card\, I\ge n+1$. In the following theorem suppose that the following conditions are fulfilled:

(*a*) *for any point* $x\in R^n$ *and any d-convex set* $S\subset R^n$ *there is*

$$d-conv(x\cup S)=\bigcup_{y\in S}\langle x,y\rangle$$

(such *d*-convexity is called *d-conical* [5])

(*b*) *for any* $S_i$ *of* $F$ *all its subsets* $S_i(x)$, $x\in S_i$ *are closed.*

Let's mention than both conditions are fulfilled in case of linear convexity in $R^n$, also for any *d*-convexity in $R^2$ [4, 6].

The following theorem is an analogue of the Helly Theorem for starshaped sets [2, Thm. 4] an its proof uses some ideas from [2] adapted for *d*-convexity.

**Theorem.** *If the intersection of every n + 1 (and less) sets of the above mentioned family* $F$ *is nonempty and d-starshaped than the intersection of the whole family is nonempty and d-starshaped.*

*Proof.* Let $S^*=\bigcap_{i\in I} S_i$. By virtue of [2, Thm. 2] $S^*\ne\varnothing$. We establish that the set $S^*$ is *d*-starshaped.

Further, we consider the sets

$$C_i=\bigcap_{x\in S^*} S_i(x),\ i\in I.$$

On the condition (*b*) of the theorem each set $C_i \subset S_i$, $i \in I$, is compact and each $C_i$, $i \in I$, is nonempty because the *d-kern* $S_i \subset C_i$. Fix some positive *k*, $k \le n+1$, and consider arbitrary sets $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ of the family $\{C_i\}_{i \in I.}$ We show that set

$$C_{i_1 \ldots i_k} = \bigcap_{j=1}^{k} C_{i_j}$$

is nonempty and *d*-starshaped.

Let $S_{i_1 \ldots i_k} = \bigcap_{j=1}^{k} S_{i_j}$, $C_{i_j} \subset S_{i_j}$, $j = \overline{1,k}$. By the hypothesis of the theorem the set $S_{i_1 \ldots i_k}$ is nonempty *d*-starshaped. Let $x \in$ *d-kern* $S_{i_1 \ldots i_k}$. Then $x \in C_{i_1 \ldots i_k}$ and as a result $C_{i_1 \ldots i_k} \neq \varnothing$. Further, select any points $x \in$ *d-kern* $S_{i_1 \ldots i_k}$, $y \in C_{i_1 \ldots i_k}$ and $z \in S^*$. Since $\langle y, z \rangle \subset C_{i_1 \ldots i_k} \subset S_{i_1 \ldots i_k}$ it follows that $\langle x, p \rangle \subset S_{i_1 \ldots i_k}$ for each point $p \in \langle y, z \rangle$. From the condition (*a*) it follows

$$d\text{-}conv\{x, y, z\} = \bigcup_{p \in \langle y, z \rangle} \langle x, p \rangle .$$

Therefore, $d-conv\{x, y, z\} \subset S_{i_1 \ldots i_k}$. Consequently, for all points *q* of $\langle x, y \rangle$ the d-segment $\langle q, z \rangle$ is contained in $S_{i_1 \ldots i_k}$. Since $z \in S^*$ was arbitrary we conclude that $\langle x, y \rangle \subset C_{i_1 \ldots i_k}$.

Thus, we show that the intersection of every *k* ($k \le n + 1$) sets of the family $\{C_i\}_{i \in I}$ is nonempty and *d*-starshapd. Then in view of [2, Thm. 2] we have $\bigcap_{i \in I} C_i \neq \varnothing$. Since

$$\bigcap_{i \in I} C_i \subset d\text{-}kern\ S^*$$

we obtain that the set $S^*$ is *d*-starshaped. The theorem is proved.

From the theorem just established we obtain, an immediate consequence, the Helly Theorem for starshaped sets proved in [2].

**Corollary 1.** *Let* $\{K_i\}_{i \in I}$ *be a family of starshaped compact sets in Euclidian space* $E^n$ *and let card* $I \ge n + 1$. *For every subfamily* $\{K_i\}_{i \in I_0}$ *of this family with card* $I_0 \le n+1$, *let the intersection* $\bigcap_{i \in I_0} K_i$ *be nonempty and starshaped. Then the intersection* $\bigcap_{i \in I} K_i$ *is nonempty and starshaped.*

Indeed in $E^n$ (when *d*-starshapedness and *d*-convexity coincides with starshapedness and linear convexity respectively (see, e.g. [3, 4]) the conditions (*a*) and (*b*) are satisfied.

**Corollary 2.** *Let* $\{N_i\}_{i \in I}$ ( *card* $I \ge 3$) *be a family of compact d-starshaped sets in* $R^2$. *If the intersection of every free (or two) sets from this family is nonempty and d-starshaped than the intersection at the whole family is also nonempty and d-starshaped.*

**References:**


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