QUEUING SYSTEMS WITH SEMI-MARKOV FLOW IN DIFFUSION APPROXIMATION SCHEMES [1]

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Se studiază schema de difuzie aproximativă asimptotic a sistemelor de aşteptare semi-Markov prin evoluţia aleatoare şi folosind operatorul de compensaţie a procesului extins Markov de reînnoire. Aceste rezultate generalizează fluxul sistemelor de aşteptare Markov şi de reînnoire.

Introduction

The queuing system (QS) of $[SM|M|1|\infty]^N$ type means that the input flow is described by a semi-Markov process, the service time is exponentially distributed, there are N servers connected by a route probability matrix. So the queuing networks is considered with a semi-Markov flow. The QS of $[SM|M|1|\infty]^N$ is considered in the series scheme with the small parameter $\varepsilon \longrightarrow 0, \varepsilon > 0$. The specific our queuing system is that series scheme is considered with phase merging procedure [3].The algorithm of diffusion approximation are established for the queuing process (QP) described of the number of claims in every node and by using the random evolution approach on the Banach space $C^3(R)$. The main tool to this end is the compensating operator of the extended Markov renewal process. We study diffusion approximation scheme for semi-Markov queuing systems by a random evolution approach and using compensating operator of the corresponding extended Markov process.Stochastic approximation of QS is a very active and interesting method to obtain numerical but also qualitative results for complex systems.

Preliminaries

The semi-Markov process $\kappa(t)$, $t \geq 0$ on the measurable phase space (E, ε) is given by the semi-Markov kernel.

$$Q(x, B, t) = G_x(t)P(x, B), x \in E, B \in \varepsilon, t \geq 0.$$

The stochastic kernel

$$P(x, B) = P\{\kappa_{n+1} \in B \mid \kappa_n = x\}, x \in E, B \in \varepsilon.$$

defines the transitions probabilities of embedded Markov chain $\kappa_n, n \geq 0$. The family of distribution functions

$$G_x(t) = P\{\Theta_{n+1} \leq t \mid \kappa_n = x\} =: P\{\Theta_x \leq t\}, x \in E$$

defines the sojourn times $\Theta_x, x \in E$. The counting process

$$\nu(t) = max\{n : \tau_n \leq t\}, t \geq 0$$

defines the number of renewal moments

$$\tau_n = \tau_{n+1} + \Theta_n, n \geq 1, \tau_0 = 0.$$

Introduce the mean values of sojourn time

$$g(x) := E\theta_x = \int_0^\infty \overline{G}_x(t)dt, \overline{G}_x(t) := 1 - G_x(t),$$

[1]This work is supported partially by Russian Foundation for Basic Research grant (RFFI) 08.820.08.09RF and grant 09.820.08.01GF of the Federal Ministry of Education and Research (BMBF) of Germany.

and the average intensities

$$q(x) = 1/g(x), x \in E.$$

The Markov renewal process (MRP)$\kappa_n \tau_n, n \geq 0$ is defined by the transition probabilities

$$P(\kappa_{n+1} \in B, \Theta_{n+1} \leq t \mid \kappa_n = x) = P(\kappa_{n+1} \in B \mid \kappa_n = x) P(\Theta_{n+1} \leq t \mid \kappa_n = x).$$

In particular case of the finite state space $\widehat{E} = \{1, 2, ..., N\}$, the semi-Markov process $\kappa(t), t \geq 0$, is defined by the semi-Markov matrix

$$Q(t) = [Q_{\kappa r}(t), \kappa, r \in \widehat{E}], Q_{\kappa r}(t) = P_{\kappa r} G_{\kappa}(t).$$

In particular case of Markov process $\kappa(t), t \geq 0$, is defined by stochastic kernel $P(x, B)$ and the intensity function $q(x), x \in E$, of exponential distribution of sojourn times Θ_x:

$$P(\Theta_{n+1} \leq t \mid \kappa_n = x) = P(\Theta_x \leq t) = 1 - e^{-q(x)t}.$$

The associated Markov process $\kappa^0(t), t \geq 0$, given by the generator

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)].$$

Markov process $\kappa^0(t), t \geq 0$ is uniformly ergodic with the stationary distribution $\pi(B), B \in \varepsilon$. Note that, the following relations takes place

$$\pi(dx) q(x) = q\rho(dx), q = \int_E \pi(dx) q(x)$$

where $\rho(B), B \in \varepsilon$ is the stationary distribution of embedded Markov chain $\kappa_n^0 = \kappa^0(\tau_n), n \geq 0$(of the Markov process $\kappa^0(t), t \geq 0$):

$$\rho(B) = \int_E \rho(dx) P(x, B), B \in \varepsilon, \rho(E) = 1.$$

The regular semi-Markov process $k^\varepsilon(t)$, $t \geq 0$ on the standard phase space (E, ε) in the series scheme, with the small series parameter $\varepsilon \to 0$ $(\varepsilon > 0)$, given by the semi-Markov kernel [1,3,4].

$$Q^\varepsilon(x, B, t) = P^\varepsilon(x, B) G_x(t), x \in E, B \in \varepsilon, t \geq 0.$$

The stochastic kernel

$$P^\varepsilon(x, B) = P(x, B) + \varepsilon P_1(x, B).$$

The stochastic kernel $P(x, B)$ is coordinated with the split phase space

$$E = \bigcup_{k=1}^{N} E_k, E_k \bigcap E_k = \emptyset, k \neq k',$$

as follows

$$P(\kappa, E_k) = \delta_k(\kappa) := \{ \begin{matrix} 1, \kappa \in E_k \\ 0, \kappa \notin E_k \end{matrix} .$$

The perturbing kernel $P_1(\kappa, B)$ provides the transition probabilities of the embedded Markov chain k_n^ε, $n \geq 0$, between classes of states E_k, $1 \leq k \leq N$, which tend to zero as $\varepsilon \to 0$. The associated Markov process is uniformly ergodic in every class $E_\kappa, \kappa \in \widehat{E}$, with the stationary distributions $\pi_\kappa(dx), \kappa \in \widehat{E}$. The corresponding embedded Markov chain $\kappa_n^0 = \kappa^0(\tau_n), n \geq 0$, is uniformly ergodic also with the stationary distributions $\rho_\kappa(dx), \kappa \in \widehat{E}$. Note that the following relations are valid:

$$\pi_\kappa(dx)q(x) = q_\kappa \rho_\kappa(dx), q_\kappa = \int\limits_{E_\kappa} \pi_\kappa(dx)q(x).$$

We assume that the merged Markov process $\widehat{\kappa}, t \geq 0$, is ergodic with the stationary distribution $\widehat{\pi} = (\widehat{\pi_\kappa}, \kappa \in \widehat{E})$. The queuing system (QS) of $[SM \mid M \mid 1 \mid \infty]^N$ type means that the input flow is described by a semi-Markov process, the service time is exponentially distributed; there are N server connected by a route probability matrix. So the queuing networks are considered with a semi-Markov input flow. The evolution of claims in the networks on $\hat{E} = \{1, 2, \ldots, N\}$ is defined by the route matrix P_0 and the intensity vector of exponential service time $\mu = (\mu_k, \; k \in \hat{E})$.

The following assumption is supposed to be valid:

A1: The queuing networks is open, that means the route matrix satisfies the condition:

$$p_{k0}^0 := 1 - \sum_{r=1}^{N} p_{kr}^0, \max_{k \in \hat{E}} p_{k0}^0 > 0.$$

A2: There exists nonnegative solution of the evolutionary equation

$$dU^0(t)/dt = C(U^0(t)), U^0(0) = u_0,$$

where the velocity vector

$$C(u) = (C_k(u), k \in \hat{E}),$$

is defined by its components

$$C_k(u) = \gamma_k(u) + \lambda_k,$$

$$\gamma_k(u) = \sum_{r=1}^{N} \mu_r u_r [p_{rk} - \delta_{rk}], \lambda_k = \hat{\pi}_k q_k$$

The queuing process in average scheme is considered in the following normalizing form:

$$U^\varepsilon(t) = \varepsilon^2 \rho^\varepsilon(t/\varepsilon^2), t \geq 0, \varepsilon > 0,$$

where $\rho^\varepsilon(t) = (\rho_k^\varepsilon(t), \; k \in \hat{E})$ is the vector with the components $\rho_k^\varepsilon(t)$ – number of claims at node $k \in \hat{E}$ at time t.

Queuing system evolution in diffusion approximation scheme

We use such processes:
$\nu^\varepsilon(t) = \varepsilon^2 \nu(t/\varepsilon^3), t \geq 0$ - normalized number claims in moment t ;
$\eta(t), t \geq 0$- MP evolution claims inside QS;
$\delta_\varepsilon(t) = \sum\limits_{n=1}^{\nu^\varepsilon(t)}, t \geq 0$ describe input flow;
$\nu_\varepsilon(t) = \eta(t) + \delta_\varepsilon(t)$.

Then QS evolution process is represented in form

$$\nu^\varepsilon(t) = \eta^\varepsilon(t) + \delta^\varepsilon(t), t \geq 0,$$

where $\eta^\varepsilon(t) = \varepsilon^2\eta(t/\varepsilon^3), \delta^\varepsilon(t) = \varepsilon^2 \sum\limits_{n=1}^{\nu(t/\varepsilon^3)} \delta(\kappa_n^\varepsilon), t \geq 0$. The normalized and centered QS process is considered in the following form

$$\zeta^\varepsilon(t) = \nu^\varepsilon(t) - \varepsilon^{-1}\rho,$$

where ρ is a unique equilibrium point of average velocity

$$C(\rho) = 0, C(\rho) = \gamma(\rho) + \lambda, \gamma(\rho) = \rho P_\mu$$

$$P_\mu = \mu^d[P_0 - I], \lambda = (\lambda_k := \widehat{\pi} q_k, k \in \widehat{E}).$$

Teorema.(Diffusion approximation) Under assumptions A1-A2 the weak convergence

$$\zeta^\varepsilon(t) \Longrightarrow \zeta^0(t)$$

as $\varepsilon \rightarrow 0$, takes place.

The limit diffusion process $\zeta^0(t), t \geq 0$ Ornstein-Uelenbeck type is defined by the generator

$$L^0\varphi(u) = \gamma(u)\varphi^{'}(u) + \frac{1}{2}B\varphi^{''}(u),$$

where

$$B =^{\widehat{\pi}} q_\kappa \widehat{R}^0_{\kappa\kappa'} q_{\kappa'} + \widehat{\pi}_{\kappa'} q_{\kappa'} \widehat{R}^0_{\kappa'\kappa} q_\kappa.$$

Here $\widehat{R}_0 = [\widehat{R}^0_{\kappa\kappa'}; \kappa, \kappa' \in \widehat{E}]$ is the potential of the merged Markov process $\widehat{\kappa}(t), t \geq 0$ on the merged phase space $\widehat{E} = \{1, 2, ..., N\}$ that is the following relation $\widehat{Q}\widehat{R}_0 = \widehat{R}_0\widehat{Q} = \widehat{\Pi} - I$, takes place.

Where the Markov process $\eta(t), t \geq 0$ describes an evolution of claims in networks, defined by the generator

$$\Gamma^\varepsilon\varphi(u) = \sum_{\kappa,r=1}^{N} \gamma_{\kappa r}(u)[\varphi(u + e_{\kappa r}) - \varphi(u)],$$

where vector of jump and intensity of jump represented by

$$e_\kappa r := e_r - e_\kappa, e_\kappa := (\delta_\kappa(l), l \in \widehat{E}), \kappa \in \widehat{E},$$

$$\gamma_\kappa r(u) = u_\kappa \mu_\kappa p^0_{kr}, \kappa = \overline{1, N}, r = \overline{0, N}, \kappa \neq r.$$

The process $\delta^\varepsilon(t), t \geq 0$, describes on input semi-Markov flow. The extended Markov renewal process (MRP)

$$\zeta_n^\varepsilon := \zeta^\varepsilon(\tau_n^\varepsilon), \kappa_n^\varepsilon := \kappa^\varepsilon(\tau_n^\varepsilon), \tau_n^\varepsilon := \varepsilon^3\tau_n, n \geq 0$$

is characterized by the compensative operator (CO) on the test-functions $\varphi(u, x), u \in R^N, x \in E$,

$$L^\varepsilon\varphi(u, x) = \varepsilon^{-3}q(x)E[\varphi(\zeta_{n+1}^\varepsilon, \kappa_{n+1}^\varepsilon) - \varphi(u, x) \mid \zeta_n^\varepsilon = u, \kappa_n^\varepsilon = x.]$$

Lema. The CO is represented in the following form:

$$L^\varepsilon\varphi(u, x) = \varepsilon^{-3}q(x)[G^\varepsilon(\rho, x)P^\varepsilon D^\varepsilon(x) - I],$$

here

$$G^\varepsilon(\rho, x) = \int_0^\infty G_x(dt)\Gamma_t^\varepsilon(\rho),$$

$\Gamma_t^\varepsilon(\rho), t \geq 0$ is the semigroup defined by the generator

$$\Gamma_\varepsilon\varphi(u) = \sum_{\kappa,r=1}^{N} \gamma_{\kappa r}(\rho + \varepsilon u)[\varphi(u + \varepsilon^2 e_{r\kappa}) - \varphi(u)],$$

$$D^{\varepsilon}(\kappa)\varphi(u) = \varphi(u + \varepsilon^2 e_{\kappa}),$$

and he operator P^{ε} defined by

$$P^{\varepsilon} = P + \varepsilon P_1,$$

$$P\varphi(x) = \int_E P(x, dy)\varphi(y), P_1\varphi(x) = \int_E P_1(x, dy)\varphi(y).$$

Lema. The generator $\Gamma_{\varepsilon}(\rho)\varphi(u)$ of MP $\nu_{\varepsilon}(t), t \geq 0 \geq$ is represented in the following form:

$$\Gamma_{\varepsilon}(\rho)\varphi(u) = \Gamma^{\varepsilon}(\rho + \varepsilon u)\varphi(u).$$

The asymptotic expansion of generator is

$$\Gamma_{\varepsilon}(\rho)\varphi(u) = \varepsilon^2\Gamma(\rho)\varphi(u) + \varepsilon^3\Gamma(u)\varphi(u) + \varepsilon^2\Theta^{\varepsilon}_{\gamma}\varphi(u)$$

where operator Γ defined by

$$\Gamma^{\varepsilon}\varphi(u) = \varepsilon^2\Gamma\varphi(u) + \varepsilon^2\Theta^{\varepsilon}_{\gamma}\varphi(u), \Gamma\varphi(u) = \gamma(u)\varphi^{'}(u) = \sum_{k=1}^{N}\gamma_k(u)\varphi^{'}_k(u).$$

Lema. The CO admits the asymptotic expansion on a test functions $\varphi(u,x) \in C^3(R^N)$ uniformly on $x \in E$ in the following form:

$$L^{\varepsilon}\varphi(u,x) = [\varepsilon^{-3}Q + \varepsilon^{-2}Q_1 + \varepsilon^{-1}C_1(\rho,x) + C_0(u,x) + \Theta^{\varepsilon}_{L}(\rho,x)]\varphi(u,x),$$

where

$$Q\varphi(x) = q(x)\int P(x,dy)[\varphi(y) - \varphi(x)]; Q_1\varphi(x) = q(x)\int P_1(x,dy)\varphi(y);$$

$$C_1(\rho,x)\varphi(u) = [\gamma(\rho) + q(x)\delta(x)]\varphi^{'}(u); C_0(u,x)\varphi(u) = [\gamma(u) + q(x)P_1\delta(x)]\varphi^{'}(u).$$

Lema. The solution of singular perturbation problem for the truncated operator

$$L^{\varepsilon}_0 = \varepsilon^{-3}Q + \varepsilon^{-2}Q_1 + \varepsilon^{-1}C_1(\rho,x) + C_0(u,x)$$

is realized on the perturbed test-function

$$\varphi^{\varepsilon}(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x) + \varepsilon^3\varphi_3(u,x)$$

in the following form

$$L^{\varepsilon}\varphi^{\varepsilon}(u,x) = L\varphi(u) + \Theta^{\varepsilon}_{L}(x)\varphi(u).$$

The limit operator L is defined by the formulae

$$L = \widehat{\Pi}\widehat{C}_1(\rho,r)R_0\widehat{C}_1(\rho,r)\widehat{\Pi} + \widehat{\Pi}\widehat{C}_0(u,r\widehat{\Pi}),$$

$$\widehat{C}_1(\rho,r) = \Pi\Pi\Pi\Pi_1(\rho,x)\Pi, \widehat{C}_0(u,r) = \Pi\Pi_0(u,x)\Pi.$$

It is worth noticing that

$$\Pi C_1(\rho,x) = \gamma(\rho) + \Pi q(x)\delta(x) = \gamma(x) + \Lambda(r), \Lambda(r) = (q_{\kappa}\delta_{\kappa}(r), \kappa, r \in \widehat{E})$$

and

$$\widehat{\Pi}\Lambda(r) = (\sum_{r=1}^{N}\pi_{\kappa}q_{\kappa}\delta_{\kappa}(r), \kappa \in E) = (\widehat{\pi}_{\kappa}q_{\kappa}, \kappa \in \widehat{E}) = \lambda.$$

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Prezentat la 30.01.2009