MULȚIMI DE ECHILIBRE STACKELBERG ÎN JOCURILE DIADICE ÎN STRATEGII MIXTE

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Catedra Matematică Aplicată

We consider the problem of determining the set of Stackelberg equilibria for dyadic games in mixed strategies. We propose an algorithm for determining the Stackelberg equilibria in dyadic games.The main results are formulated and explained. A procedure for the equilibrium set determining is presented. It is applied to solve illustration examples.

În lucrare se cercetează noțiunea de echilibru Stackelberg prin detalierea/particularizarea tezelor teoretice din [1-5] în cazul jocurilor diadice.

Se consideră jocul

$$\Gamma = \left\langle \{1,2\}, \mathbf{X}, \mathbf{Y}, f_1(x,y), f_2(x,y) \right\rangle,$$

unde:

- $\{1,2\}$ este mulțimea de jucători,
- $\mathbf{X}, \mathbf{Y}$ sunt mulțimile de strategii ale jucătorulor *1* şi *2*, respectiv,

$$\mathbf{X} = \left\{ (x_1, x_2) : x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0 \right\},$$
$$\mathbf{Y} = \left\{ (y_1, y_2) : y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0 \right\},$$

- $f_1(x,y), f_2(x,y)$ sunt funcțiile de câştig ale jucătorilor *1* şi *2*, respectiv, definite pe produsul cartezian $\mathbf{X} \times \mathbf{Y}$.

Se presupune, fără a pierde din generalitate, că toți jucătorii îşi maximizează valorile funcțiilor de câştig.

Fie $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ şi $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ matricele funcțiilor de câştig. Pentru primul jucător funcția de câştig poate fi scrisă după cum urmează:

$$f_1(x,y) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{vmatrix} x_1 = x \geq 0, & x_2 = 1 - x_1 \\ y_1 = y \geq 0, & y_2 = 1 - y_1 \end{vmatrix},$$

$$f_1(x,y) = \left[(a_{11} - a_{12} - a_{21} + a_{22}) y + (a_{12} - a_{22}) \right] x + (a_{21} - a_{22}) y + a_{22};$$

pentru al doilea:

$$f_2(x,y) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{vmatrix} x_1 = x \geq 0, & x_2 = 1 - x_1 \\ y_1 = y \geq 0, & y_2 = 1 - y_1 \end{vmatrix},$$

$$f_2(x,y) = \left[(b_{11} - b_{12} - b_{21} + b_{22}) x + (b_{21} - b_{22}) \right] y + (b_{12} - b_{22}) x + b_{22}.$$

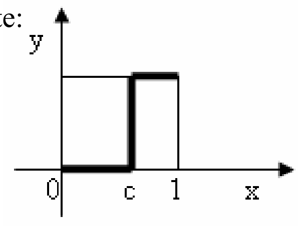
Se construieşte aplicația de tip cel mai bun răspuns al jucătorului doi. Dacă notăm:

$$\beta(x) = (b_{11} - b_{12} - b_{21} + b_{22}) x + (b_{21} - b_{22}),\ \eta = b_{11} - b_{12} - b_{21} + b_{22},\ \mu = b_{21} - b_{22},$$

atunci putem evidenția 9 cazuri.

Cazul I. Dacă $\eta > 0, \mu < 0, \eta > -\mu$, atunci graficul jucătorului doi este:

$$Gr_2 = [0,1] \times [0,1] \cap \begin{cases} [0,1], & \textit{dacă} \quad x = c, \\ 1, & \textit{dacă} \quad c < x \leq 1, \\ 0, & \textit{dacă} \quad 0 \leq x < c, \end{cases}$$



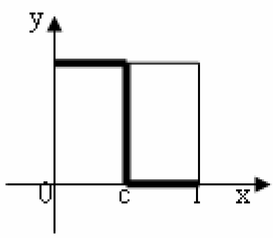
iar mulțimea de echilibre Stackelberg este:

$$\mathbf{SE}=\begin{cases}(1,1), & \text{dacă } a_{11}>f_1,f_2,a_{22},\\ \left[(c,1),(1,1)\right], & \text{dacă } a_{11}=f_1>f_2,a_{22},\\ \left\{(c,0),(1,1)\right\}, & \text{dacă } a_{11}=f_2>f_1,a_{22},\\ \left\{(0,0),(1,1)\right\}, & \text{dacă } a_{11}=a_{22}>f_2,f_1,\\ \left\{(0,0)\right\}\cup\left[(c,1),(1,1)\right], & \text{dacă } a_{11}=a_{22}=f_1>f_2,\\ \left\{(0,0)\right\}\cup\left[(c,0),(c,1)\right]\cup\left[(c,1),(1,1)\right], & \text{dacă } a_{11}=a_{22}=f_1<f_2,\\ \left\{(1,1)\right\}\cup\left[(0,0),(c,0)\right], & \text{dacă } a_{11}=a_{22}=f_2>f_1,\\ \left\{(1,1)\right\}\cup\left[(0,0),(c,0)\right]\cup\left[(c,0),(c,1)\right], & \text{dacă } a_{11}=a_{22}=f_2<f_1,\\ \left[(c,0),(c,1)\right]\cup\left[(c,1),(1,1)\right], & \text{dacă } a_{11}=f_2=f_1>a_{22} \text{ sau } f_2>a_{11}=f_1>a_{22},\\ (0,0), & \text{dacă } a_{22}>f_1,f_2,a_{11},\\ \left\{(0,0),(c,1)\right\}, & \text{dacă } a_{22}=f_1>f_2,a_{11},\\ \left[(0,0),(c,0)\right], & \text{dacă } a_{22}=f_2>f_1,a_{11},\\ \left[(0,0),(c,0)\right]\cup\left[(c,0),(c,1)\right], & \text{dacă } a_{22}=f_1=f_2>a_{11} \text{ sau } f_1>a_{22}=f_2>a_{11},\\ (c,1), & \text{dacă } f_1>f_2,a_{11},a_{22} \text{ și } a_{11}\neq a_{22},\\ \left[(c,0),(c,1)\right], & \text{dacă } f_1=f_2>a_{11},a_{22},\\ (c,0), & \text{dacă } f_2>f_1,a_{11},a_{22} \text{ și } a_{11}\neq a_{22},\\ \left[(0,0),(c,0)\right]\cup\left[(c,0),(c,1)\right]\cup\left[(c,1),(1,1)\right], & \text{dacă } f_1=f_2=a_{11}=a_{22},\\ \left\{(1,1)\right\}\cup\left[(c,0),(c,1)\right], & \text{dacă } f_1>a_{11}=f_2>a_{22},\\ \left\{(0,0)\right\}\cup\left[(c,0),(c,1)\right], & \text{dacă } f_2>a_{22}=f_1>a_{11},\\ \left\{(0,0),(1,1)\right\}\cup\left[(c,0),(c,d)\right], & \text{dacă } f_2>a_{11}=a_{22}>f_1,\\ \left\{(0,0),(1,1)\right\}\cup\left[(c,d),(c,1)\right], & \text{dacă } f_1>a_{11}=a_{22}>f_2,\end{cases}$$

unde $f_1=(a_{11}-a_{21})c+a_{21}, f_2=(a_{12}-a_{22})c+a_{22}, d=\dfrac{c(a_{22}-a_{12})}{a_{21}-a_{22}+c(2a_{22}-a_{12}-a_{21})}$.

Cazul II. Dacă $\eta<0,\mu>0,-\eta>\mu$, atunci:

$$Gr_2=[0,1]\times[0,1]\cap\begin{cases}[0,1], & \text{dacă} \quad x=c\\ 0, & \text{dacă} \quad c<x\le 1\\ 1, & \text{dacă} \quad 0\le x<c,\end{cases}$$



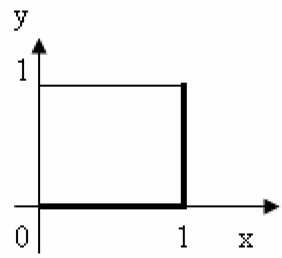
iar mulțimea de echilibre Stackelberg este:

$$
\mathbf{SE}=\begin{cases}
(0,1), & \textit{dacă } a_{21}>f_1,f_2,a_{12},\\
[(0,1),(c,1)], & \textit{dacă } a_{21}=f_1>f_2,a_{12},\\
\{(c,0),(0,1)\}, & \textit{dacă } a_{21}=f_2>f_1,a_{12},\\
\{(0,1),(1,0)\}, & \textit{dacă } a_{21}=a_{12}>f_2,f_1,\\
\{(1,0)\}\cup[(0,1),(c,1)], & \textit{dacă } a_{21}=a_{12}=f_1>f_2,\\
\{(1,0)\}\cup[(c,0),(c,1)]\cup[(c,1),(0,1)], & \textit{dacă } a_{21}=a_{12}=f_1<f_2,\\
\{(0,1)\}\cup[(c,0),(1,0)], & \textit{dacă } a_{21}=a_{12}=f_2>f_1,\\
\{(0,1)\}\cup[(c,1),(c,0)]\cup[(c,0),(1,0)], & \textit{dacă } a_{21}=a_{12}=f_2<f_1,\\
[(0,1),(c,1)]\cup[(c,1),(c,0)], & \textit{dacă } a_{21}=f_2=f_1>a_{12} \textit{ sau } f_2>a_{21}=f_1>a_{12},\\
(1,0), & \textit{dacă } a_{12}>f_1,f_2,a_{21},\\
\{(1,0),(c,1)\}, & \textit{dacă } a_{12}=f_1>f_2,a_{21},\\
[(c,0),(1,0)], & \textit{dacă } a_{12}=f_2>f_1,a_{21},\\
[(c,1),(c,0)]\cup[(c,0),(1,0)], & \textit{dacă } a_{12}=f_1=f_2>a_{21} \textit{ sau } f_1>a_{12}=f_2>a_{21},\\
(c,1), & \textit{dacă } f_1>f_2,a_{21},a_{12} \textit{ și } a_{21}\neq a_{12},\\
[(c,0),(c,1)], & \textit{dacă } f_1=f_2>a_{21},a_{12},\\
(c,0), & \textit{dacă } f_2>f_1,a_{21},a_{12} \textit{ și } a_{21}\neq a_{12},\\
[(0,1),(c,1)]\cup[(c,1),(c,0)]\cup[(c,0),(1,0)], & \textit{dacă } f_1=f_2=a_{21}=a_{12},\\
\{(0,1)\}\cup[(c,0),(c,1)], & \textit{dacă } f_1>a_{21}=f_2>a_{12},\\
\{(1,0)\}\cup[(c,0),(c,1)], & \textit{dacă } f_2>a_{12}=f_1>a_{21},\\
\{(0,1),(1,0)\}\cup[(c,0),(c,d)], & \textit{dacă } f_2>a_{21}=a_{12}>f_1,\\
\{(0,1),(1,0)\}\cup[(c,d),(c,1)], & \textit{dacă } f_1>a_{21}=a_{12}>f_2,
\end{cases}
$$

unde $f_1=(a_{11}-a_{21})c+a_{21}, f_2=(a_{12}-a_{22})c+a_{22}, d=\dfrac{a_{12}-a_{22}+c(a_{22}-a_{12})}{a_{12}-a_{22}+c(a_{11}-2a_{12}+a_{22})}$.

Cazul III. Dacă $\eta>0,\mu<0,\eta=-\mu$, atunci:

$$Gr_2=[0,1]\times[0,1]\cap\begin{cases}[0,1], & \textit{dacă } x=1\\ 0, & \textit{dacă } 0\le x\le 1,\end{cases}$$



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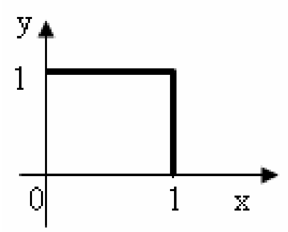
$$\mathbf{SE}=\begin{cases}(0,0), & dacă\ a_{22}>a_{11},a_{12},\\ (1,0), & dacă\ a_{12}>a_{11},a_{22},\\ (1,1), & dacă\ a_{11}>a_{22},a_{12},\\ \{(0,0),(1,1)\}, & dacă\ a_{22}=a_{11}>a_{12},\\ [(0,0),(1,0)], & dacă\ a_{22}=a_{12}>a_{11},\\ [(0,0),(1,0)]\cup[(1,0),(1,1)], & dacă\ a_{22}=a_{12}=a_{11}\ sau\ a_{22}=a_{12}<a_{11},\\ \{(0,0)\}\cup[(1,0),(1,1)], & dacă\ a_{22}=a_{11}<a_{12},\\ [(1,0),(1,1)], & dacă\ a_{12}=a_{11}>a_{22},\\ \{(0,0)\}\cup[(1,d),(1,1)], & dacă\ a_{11}>a_{22},\ a_{11}>a_{12},\ a_{22}>a_{12},\\ \{(0,0)\}\cup[(1,0),(1,d)], & dacă\ a_{12}>a_{22},\ a_{12}>a_{11},\ a_{22}>a_{11},\end{cases}$$

unde $d=\dfrac{a_{22}-a_{12}}{a_{11}-a_{12}}$.

Cazul IV. Dacă $\eta<0,\mu>0,-\eta=\mu$, atunci:

$$Gr_2=[0,1]\times[0,1]\cap\begin{cases}[0,1],\ dacă\ x=1\\ 1,\ dacă\ 0\le x\le 1\end{cases}$$



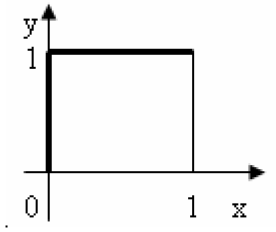
şi

$$\mathbf{SE}=\begin{cases}(0,1), & dacă\ a_{21}>a_{11},a_{12},\\ (1,0), & dacă\ a_{12}>a_{11},a_{21},\\ (1,1), & dacă\ a_{11}>a_{21},a_{12},\\ \{(0,1),(1,0)\}, & dacă\ a_{21}=a_{12}>a_{11},\\ [(0,1),(1,1)], & dacă\ a_{21}=a_{11}>a_{12},\\ [(0,1),(1,1)]\cup[(1,1),(1,0)], & dacă\ a_{21}=a_{12}=a_{11}\ sau\ a_{21}=a_{11}<a_{12},\\ \{(0,1)\}\cup[(1,0),(1,1)], & dacă\ a_{21}=a_{12}<a_{11},\\ [(1,0),(1,1)], & dacă\ a_{12}=a_{11}>a_{21},\\ \{(0,1)\}\cup[(1,d),(1,1)], & dacă\ a_{11}>a_{21},\ a_{11}>a_{12},\ a_{21}>a_{12},\\ \{(0,1)\}\cup[(1,0),(1,d)], & dacă\ a_{12}>a_{21},\ a_{12}>a_{11},\ a_{21}>a_{11},\end{cases}$$

unde $d=\dfrac{a_{21}-a_{12}}{a_{11}-a_{12}}$.

Cazul V. Dacă $\eta>0,\mu=0$, atunci:

$$Gr_2=[0,1]\times[0,1]\cap\begin{cases}[0,1],\ dacă\ x=0\\ 1,\ dacă\ 0\le x\le 1\end{cases}$$



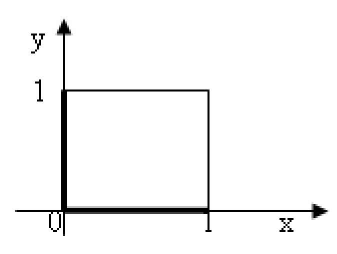
iar

$$\mathbf{SE}=\begin{cases}(1,1), & dacă\ a_{11}>a_{21},a_{22},\\ (0,0), & dacă\ a_{22}>a_{11},a_{21},\\ (0,1), & dacă\ a_{21}>a_{22},a_{11},\\ \{(0,0),(1,1)\}, & dacă\ a_{22}=a_{11}>a_{21},\\ [(0,1),(1,1)], & dacă\ a_{21}=a_{11}>a_{22},\\ [(0,0),(0,1)]\cup[(0,1),(1,1)], & dacă\ a_{21}=a_{22}=a_{11}\ sau\ a_{21}=a_{11}<a_{22},\\ \{(1,1)\}\cup[(0,0),(0,1)], & dacă\ a_{11}=a_{22}<a_{21},\\ [(0,0),(0,1)], & dacă\ a_{22}=a_{21}>a_{11},\\ \{(1,1)\}\cup[(0,d),(0,1)], & dacă\ a_{21}>a_{11},\ a_{21}>a_{22},\ a_{11}>a_{22},\\ \{(1,1)\}\cup[(0,0),(0,d)], & dacă\ a_{22}>a_{11},\ a_{22}>a_{21},\ a_{11}>a_{21},\end{cases}$$

unde $d=\dfrac{a_{11}-a_{22}}{a_{21}-a_{22}}$.

Cazul VI. Dacă $\eta<0,\mu=0$, atunci:

$$Gr_2=[0,1]\times[0,1]\cap\begin{cases}[0,1],\ dacă\ x=0\\ 0,\ dacă\ 0\le x\le 1\end{cases}$$



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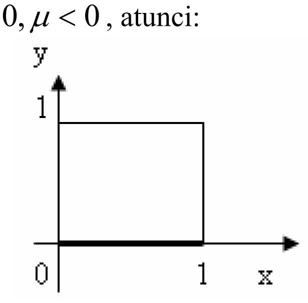
$$\mathbf{SE}=\begin{cases}(1,0), & dacă\ a_{12}>a_{21},a_{22},\\ (0,0), & dacă\ a_{22}>a_{12},a_{21},\\ (0,1), & dacă\ a_{21}>a_{22},a_{12},\\ \{(0,1),(1,0)\}, & dacă\ a_{21}=a_{12}>a_{22},\\ [(0,0),(1,0)], & dacă\ a_{22}=a_{12}>a_{21},\\ [(0,1),(0,0)]\cup[(0,0),(1,0)], & dacă\ a_{21}=a_{22}=a_{12}\ sau\ a_{22}=a_{12}<a_{21},\\ \{(1,0)\}\cup[(0,0),(0,1)], & dacă\ a_{21}=a_{12}<a_{22},\\ [(0,0),(0,1)], & dacă\ a_{22}=a_{21}>a_{12},\\ \{(1,0)\}\cup[(0,d),(0,1)], & dacă\ a_{21}>a_{12},\ a_{21}>a_{22},\ a_{12}>a_{22},\\ \{(1,0)\}\cup[(0,0),(0,d)], & dacă\ a_{22}>a_{12},\ a_{22}>a_{21},\ a_{12}>a_{21},\end{cases}$$

unde $d=\dfrac{a_{12}-a_{22}}{a_{21}-a_{22}}$.

Cazul VII. Dacă $\eta>0,\mu<0,\eta<-\mu$ sau $\eta<0,\mu<0$ sau $\eta=0,\mu<0$, atunci:

$$Gr_2=[0,1]\times[0,1]\cap\{0,\ dacă\ 0\le x\le 1,$$

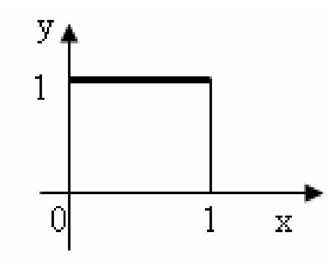


iar

$$\mathbf{SE}=\begin{cases}(0,0), & dacă\ a_{22}>a_{12},\\ (1,0), & dacă\ a_{22}<a_{12},\\ [(0,0),(1,0)], & dacă\ a_{22}=a_{12}.\end{cases}$$

Cazul VIII. Dacă $\eta < 0, \mu > 0, -\eta < \mu$ sau $\eta > 0, \mu > 0$ sau $\eta = 0, \mu > 0$, atunci:

$$Gr_2 = [0,1]\times[0,1]\cap\{1,\ dac\breve{a}\ 0 \le x \le 1,$$



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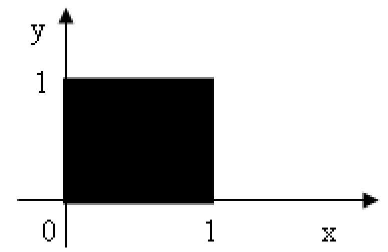
$$\mathbf{SE} = \begin{cases} (0,1), & dac\breve{a}\ a_{21} > a_{11}, \\ (1,1), & dac\breve{a}\ a_{21} < a_{11}, \\ [(0,1),(1,1)], & dac\breve{a}\ a_{21} = a_{11}. \end{cases}$$

Cazul IX. Dacă $\eta = 0, \mu = 0$, atunci:

$$Gr_2 = [0,1]\times[0,1],$$



iar

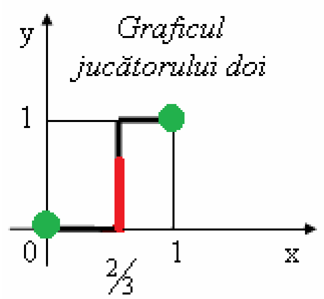
$$\mathbf{SE} = \begin{cases} (1,1), & dac\breve{a}\ a_{11} > a_{12}, a_{21}, a_{22}, \\ (0,1), & dac\breve{a}\ a_{21} > a_{12}, a_{11}, a_{22}, \\ (1,0), & dac\breve{a}\ a_{12} > a_{11}, a_{21}, a_{22}, \\ (0,0), & dac\breve{a}\ a_{22} > a_{12}, a_{21}, a_{11}, \\ [(0,0),(0,1)], & dac\breve{a}\ a_{21} = a_{22} > a_{11}, a_{12}, \\ [(0,1),(1,1)], & dac\breve{a}\ a_{21} = a_{11} > a_{22}, a_{12}, \\ [(0,0),(1,0)], & dac\breve{a}\ a_{12} = a_{22} > a_{11}, a_{21}, \\ [(1,0),(1,1)], & dac\breve{a}\ a_{11} = a_{12} > a_{21}, a_{22}, \\ \{(0,0),(1,1)\}, & dac\breve{a}\ a_{11} = a_{22} > a_{21}, a_{12}, \\ \{(0,1),(1,0)\}, & dac\breve{a}\ a_{21} = a_{12} > a_{11}, a_{22}, \\ [(1,1),(1,0)]\cup[(1,0),(0,0)], & dac\breve{a}\ a_{11} = a_{12} = a_{22} > a_{21}, \\ [(1,0),(1,1)]\cup[(1,1),(0,1)], & dac\breve{a}\ a_{11} = a_{12} = a_{21} > a_{22}, \\ [(1,1),(0,1)]\cup[(0,1),(0,0)], & dac\breve{a}\ a_{11} = a_{21} = a_{22} > a_{12}, \\ [(0,1),(0,0)]\cup[(0,0),(1,0)], & dac\breve{a}\ a_{12} = a_{21} = a_{22} > a_{11}, \\ [0,1]\times[0,1], & dac\breve{a}\ a_{11} = a_{12} = a_{21} = a_{22}. \end{cases}$$

Exemplul 1.

Fie $A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$ şi $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ matricele funcțiilor de câștig.

$f_1(x,y) = 2x - 2y + 1,\ f_2(x,y) = (3x-2)y - x + 2.$

Deoarece *3>0* şi *-2<0*, $c = \frac{2}{3}$, avem cazul I.I.a) $\Rightarrow$ graficul jucătorului II este reprezentat în partea dreaptă.



Revenim la jucătorul I : $y=0 \Rightarrow \underset{(x,0)\in Gr_2}{Arg\max} f_1(x,0)=2x+1=\frac{7}{3},\ x=\frac{2}{3}$,

$y=1 \Rightarrow \underset{(x,1)\in Gr_2}{Arg\max} f_1(x,1)=2x-1=1,\ x=1$,

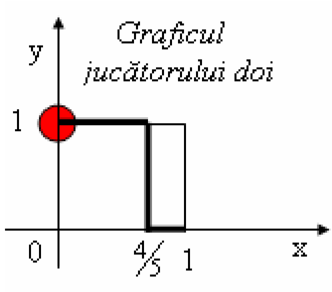
$y\in(0,1) \Rightarrow x=\frac{2}{3},\ \underset{(x,y)\in Gr_2}{Arg\max} f_1\left(\frac{2}{3},y\right)=-2y+\frac{7}{3}=\frac{7}{3},\ y=0$.

Strategia pentru care valoarea funcției de câștig este mai mare este $\left(\frac{2}{3},0\right)$ cu câștigul $\left(\frac{7}{3},\frac{4}{3}\right)$. Conform algoritmului, mulțimea de echilibre Stackelberg este reprezentată pe grafic cu culoare roșie.

Exemplul 2.

Fie $A=\begin{bmatrix}2 & 3\\ 4 & 1\end{bmatrix}$ și $B=\begin{bmatrix}5 & 6\\ 3 & -1\end{bmatrix}$ matricele funcțiilor de câștig.

$f_1(x,y)=(-4y+2)x+3y+1,\ f_2(x,y)=(-5x+4)y+7x-1.$



Deoarece *-5<0* și *4 > 0,* $c=\frac{2}{3}$, avem cazul I.I.b) $\Rightarrow$ graficul jucătorului II este reprezentat în partea dreaptă.

Revenim la jucătorul I : $y=0 \Rightarrow \underset{(x,0)\in Gr_2}{Arg\max} f_1(x,0)=2x+1=3,\ x=1$,

$y=1 \Rightarrow \underset{(x,1)\in Gr_2}{Arg\max} f_1(x,1)=-2x+4=4,\ x=0$,

$y\in(0,1) \Rightarrow x=\frac{4}{5},\ \underset{(x,y)\in Gr_2}{Arg\max} f_1\left(\frac{4}{5},y\right)=-\frac{1}{5}y+\frac{13}{5}=\frac{13}{5}=2,6,\ y=0$.

Strategia pentru care valoarea funcției de câștig este mai mare este $(0,1)$ cu câștigul $(4,3)$.

Cazurile evidențiate *supra* permit să se elaboreze un procedeu de aflare a mulțimilor de echilibre Stackelberg. Procedeul a fost programat în Mathematica 7 și publicat în Wolfram Demonstration Project [6]. *Infra* urmează codul programului de aflare a mulțimii de echilibre Stackelber în jocul diadic în strategii mixte.

```
k=Thickness[0.02];
r=0.01;
sq=Graphics[{ColorData["Legacy","Bisque"],Polygon[{{0,0},{0,1},{1,1},{1,0}}],
Axes→ True
,k,Line[{{0,0},{0,1},{1,1},{1,0},{0,0}}],Disk[{0,0},r],Disk[{0,1},r],Disk
[{1,1},r],Disk[{1,0},r]}];
sr=Graphics[{RGBColor[.25,.43,.82],Polygon[{{0,0},{0,1},{1,1},{1,0}}],k,
Line[{{0,0},{0,1},{1,1},{1,0},{0,0}}],Disk[{0,0},r],Disk[{0,1},r],Disk
[{1,1},r],Disk[{1,0},r]}];

A=Graphics[{RGBColor[.6,.73,.36],Disk[{0,0},r]}];
B=Graphics[{RGBColor[.6,.73,.36],Disk[{0,1},r]}];
P=Graphics[{RGBColor[.6,.73,.36],Disk[{1,1},r]}];
Q=Graphics[{RGBColor[.6,.73,.36],Disk[{1,0},r]}];

AB=Graphics[{k,RGBColor[.25,.43,.82],Line[{{0,0},{0,1}}]}];
BP=Graphics[{k,RGBColor[.25,.43,.82],Line[{{0,1},{1,1}}]}];
PQ=Graphics[{k,RGBColor[.25,.43,.82],Line[{{1,1},{1,0}}]}];
QA=Graphics[{k,RGBColor[.25,.43,.82],Line[{{1,0},{0,0}}]}];

AR=Graphics[{RGBColor[.49,0,0],Disk[{0,0},r]}];
BR=Graphics[{RGBColor[.49,0,0],Disk[{0,1},r]}];
PR=Graphics[{RGBColor[.49,0,0],Disk[{1,1},r]}];
QR=Graphics[{RGBColor[.49,0,0],Disk[{1,0},r]}];
```

```
ABR=Graphics[{k,RGBColor[.49,0,0],Line[{{0,0},{0,1}}]}];
BPR=Graphics[{k,RGBColor[.49,0,0],Line[{{0,1},{1,1}}]}];
PQR=Graphics[{k,RGBColor[.49,0,0],Line[{{1,1},{1,0}}]}];
QAR=Graphics[{k,RGBColor[.49,0,0],Line[{{1,0},{0,0}}]}];

GR2[a0_List,b0_List]:=Module[{m1=a0,m2=b0,s,t,c,points,lines,SE,g2},
α=m1[[1,1]];β=m1[[1,2]];γ=m1[[2,1]];δ=m1[[2,2]];
s=m2[[1,1]]-m2[[1,2]]-m2[[2,1]]+m2[[2,2]]; t=m2[[2,1]]-m2[[2,2]];
If[s≠0,c=(m2[[2,2]]-m2[[2,1]])/s];
points:={};
lines:={};

VW=Graphics[{k,RGBColor[.25,.43,.82],Line[{{c,0},{c,1}}]}];
WP=Graphics[{k,RGBColor[.25,.43,.82],Line[{{c,1},{1,1}}]}];
AV=Graphics[{k,RGBColor[.25,.43,.82],Line[{{0,0},{c,0}}]}];
BW=Graphics[{k,RGBColor[.25,.43,.82],Line[{{0,1},{c,1}}]}];
VQ=Graphics[{k,RGBColor[.25,.43,.82],Line[{{c,0},{1,0}}]}];

VR=Graphics[{RGBColor[.49,0,0],Disk[{c,0},r]}];
WR=Graphics[{RGBColor[.49,0,0],Disk[{c,1},r]}];
VWR=Graphics[{k,RGBColor[.49,0,0],Line[{{c,0},{c,1}}]}];
WPR=Graphics[{k,RGBColor[.49,0,0],Line[{{c,1},{1,1}}]}];
AVR=Graphics[{k,RGBColor[.49,0,0],Line[{{0,0},{c,0}}]}];
BWR=Graphics[{k,RGBColor[.49,0,0],Line[{{0,1},{c,1}}]}];
VQR=Graphics[{k,RGBColor[.49,0,0],Line[{{c,0},{1,0}}]}];

If[s>0&&t<0&&(s>-t),

  f1=(α-γ)*c+γ; f2=( β-δ)*c+δ;
  If[α>f1&&α>f2&&α>δ,g2:=Show[sq,AV,VW,WP,PR]];
  If[α==f1&&α>f2&&α>δ,g2:=Show[sq,AV,VW,WPR]];
  If[α==f2&&α>f1&&α>δ,g2:=Show[sq,AV,VW,WP,PR,VR]];
  If[α==f1&&f1==f2&&f2>δ,g2:=Show[sq,AV,VWR,WPR]];
  If[α==δ&&α>f1&&α>f2,g2:=Show[sq,AV,VW,WP,PR,AR]];
  If[α==δ&&α<f1&&α>f2,
  g2:=Show[sq,AV,VW,WP,P,A,Graphics[{k,RGBColor[.49,0,0],Line[{{c,(c*δ-
     *β)/(γ-δ+c*(2*δ-β-γ))},{c,1}}]}]],
  If[α==f2&&α<f1&&α>δ,g2:=Show[sq,AV,WP,VWR,P],
  If[f1>α&&f1>f2&&f1>δ,g2:=Show[sq,AV,VW,WP,WR]]]];
  If[α==δ&&α>f1&&α<f2,
  g2:=Show[sq,AV,VW,WP,P,A,Graphics[{k,RGBColor[.49,0,0],Line[{{c,0},{c,
     (c*δ-c*β)/(γ-δ+c*(2*δ-β-γ))}}]}]],
  If[α==f1&&α<f2&&α>δ,g2:=Show[sq,AV,VWR,WPR],
  If[f2>α&&f2>f1&&f2>δ,g2:=Show[sq,AV,VW,WP,VR]]]];
  If[α==δ&&δ==f1&&f1>f2,g2:=Show[sq,AV,VW,WPR,AR]];
  If[α==δ&&δ==f1&&f1<f2,g2:=Show[sq,AV,VWR,WPR,A]];
  If[α==δ&&δ==f2&&f2>f1,g2:=Show[sq,VW,WP,AVR,PR]];
  If[α==δ&&δ==f2&&f2<f1,g2:=Show[sq,WP,AVR,VWR,P]];
  If[α==δ&&δ==f2&&f2==f1,g2:=Show[sq,AVR,VWR,WPR]];
  If[δ>f1&&δ>f2&&δ>α,g2:=Show[sq,AV,VW,WP,AR]];
  If[δ==f1&&δ>f2&&δ>α,g2:=Show[sq,AV,VW,WP,AR,WR]];
  If[δ>f1&&δ==f2&&δ>α,g2:=Show[sq,AVR,VW,WP]];
  If[δ==f1&&δ<f2&&α<δ,g2:=Show[sq,AV,WP,VWR,A]];

If[(δ==f2&&δ<f1&&α<δ)||(δ==f2&&f2==f1&&f1>α),g2:=Show[sq,WP,AVR,VWR]];
     If[f1>α&&f1==f2&&f1>δ,g2:=Show[sq,AV,WP,VWR]];
```

```
  points:={{1,1}}/;α>f1&&α>f2&&α>δ;
  lines:={{c,1},{1,1}}/;α==f1&&α>f2&&α>δ;
  points:={{1,1},{c,0}}/;α==f2&&α>f1&&α>δ;
  lines:={{c,0},{c,1},{1,1}}/;α==f1&&α<f2&&α>δ;
  lines:={{c,0},{c,1},{1,1}}/;α==f1&&f1==f2&&f2>δ;
  lines:={{c,0},{c,1}}/;α==f2&&α<f1&&α>δ;
  points:={{1,1},{0,0}}/;α==δ&&α>f1&&α>f2;
  lines:={{c,(c*δ-c*β)/(γ-δ+c*(2*δ-β-γ))},{c,1}}/;α==δ&&α<f1&&α>f2;
  points:={{c,1}}/;f1>α&&f1>f2&&f1>δ&&α!=δ&&α≠f2;
  points:={{c,1}}/;f1>α&&f1>f2&&f1>δ&&α==δ;
  points:={{c,1}}/;f1>α&&f1>f2&&f1>δ&&α==f2;
  lines:={{c,0},{c,(c*δ-c*β)/(γ-δ+c*(2*δ-β-γ))}}/;α==δ&&α>f1&&α<f2;
  points:={{c,0}}/;f2>α&&f2>f1&&f2>δ&&α!=δ&&α≠f1;
  points:={{c,0}}/;f2>α&&f2>f1&&f2>δ&&α==δ;
  points:={{c,0}}/;f2>α&&f2>f1&&f2>δ&&α==f1;
  points:={{0,0}}/;α==δ&&δ==f1&&f1>f2;
  lines:={{c,1},{1,1}}/;α==δ&&δ==f1&&f1>f2;
  lines:={{c,0},{c,1},{1,1}}/;α==δ&&δ==f1&&f1<f2;
  points:={{1,1}}/;α==δ&&δ==f2&&f2>f1;
  lines:={{0,0},{c,0}}/;α==δ&&δ==f2&&f2>f1;
  lines:={{0,0},{c,0},{c,1}}/;α==δ&&δ==f2&&f2<f1;
  lines:={{0,0},{c,0},{c,1},{1,1}}/;α==δ&&δ==f2&&f2==f1;
  points:={{0,0}}/;δ>f1&&δ>f2&&δ>α;
  points:={{0,0},{c,1}}/;δ==f1&&δ>f2&&δ>α;
  lines:={{0,0},{c,0}}/;δ>f1&&δ==f2&&δ>α;
  lines:={{c,0},{c,1}}/;δ==f1&&δ<f2&&α<δ;
  lines:={{0,0},{c,0},{c,1}}/;δ==f2&&δ<f1&&α<δ;
  lines:={{0,0},{c,0},{c,1}}/;δ==f2&&f2==f1&&f1>α;
  lines:={{c,0},{c,1}}/;f1>α&&f1==f2&&f1>δ
]
  If[s<0&&t>0&&(-s>t),
  f1=(α-γ)*c+γ; f2=( β-δ)*c+δ;
  If[γ>f1&&γ>f2&&γ>β,g2:=Show[sq,BW,VW,VQ,BR]];
  If[γ==f1&&γ>f2&&γ>β,g2:=Show[sq,VW,VQ,BWR]];
  If[γ>f1&&γ==f2&&γ>β,g2:=Show[sq,BW,VW,VQ,BR,VR]];
  If[γ==f1&&f1==f2&&f2>β,g2:=Show[sq,VQ,BWR,VWR]];
  If[γ==β&&γ>f1&&γ>f2,g2:=Show[sq,BW,VW,VQ,BR,QR]];
  If[γ==β&&γ<f1&&γ>f2,
  g2:=Show[sq,BW,VW,VQ,B,Q,Graphics[{k,RGBColor[.49,0,0],Line[{{c,(β-δ-
     c*β+c*δ)/(β-δ+c*(α-2*β+δ))},{c,1}}]}]],
  If[γ==f2&&γ<f1&&γ>β,g2:=Show[sq,BW,VQ,VWR,B],
  If[γ==β&&γ>f1&&γ<f2,
  g2:=Show[sq,BW,VW,VQ,B,Q,Graphics[{k,RGBColor[.49,0,0],Line[{{c,0},{c,
     (β-δ-c*β+c*δ)/(β-δ+c*(α-2*β+δ))}}]}]],
  If[γ==f1&&γ<f2&&γ>β,g2:=Show[sq,VQ,BWR,VWR],
  If[f2>γ&&f2>f1&&f2>β,g2:=Show[sq,BW,VW,VQ,VR]]]];
  If[γ==β&&β==f1&&f1>f2,g2:=Show[sq,VW,VQ,BWR,QR]];
  If[γ==β&&β==f1&&f1<f2,g2:=Show[sq,VQ,BWR,VWR,Q]];
  If[γ==β&&β==f2&&f2>f1,g2:=Show[sq,BW,VW,VQR,BR]];
  If[γ==β&&β==f2&&f2<f1,g2:=Show[sq,BW,VWR,VQR,B]];
  If[γ==β&&β==f2&&f2==f1,g2:=Show[sq,BWR,VWR,VQR]];
  If[β>f1&&β>f2&&β>γ,g2:=Show[sq,BW,VW,VQ,QR]];
```

```
  If[β==f1&&β>f2&&β>γ,g2:=Show[sq,BW,VW,VQ,WR,QR]];
  If[β>f1&&β==f2&&β>γ,g2:=Show[sq,BW,VW,VQR]];
  If[β==f1&&β<f2&&γ<β,g2:=Show[sq,BW,VQ,VWR, Q]];
  If[(β==f2&&β<f1&&γ<β)||(β==f2&&f2==f1&&f1>γ),g2:=Show[sq,BW,VWR,VQR]];
  If[f1>γ&&f1==f2&&f1>β,g2:=Show[sq,BW,VQ,VWR]];

  points:={{0,1}}/;γ>f1&&γ>f2&&γ>β;
  lines:={{0,1},{c,1}}/;γ==f1&&γ>f2&&γ>β;
  points:={{0,1},{c,0}}/;γ>f1&&γ==f2&&γ>β;
  lines:={{0,1},{c,1},{c,0}}/;γ==f1&&γ<f2&&γ>β;
  lines:={{0,1},{c,1},{c,0}}/;γ==f1&&f1==f2&&f2>β;
  lines:={{c,1},{c,0}}/;γ==f2&&γ<f1&&γ>β;
  points:={{0,1},{1,0}}/;γ==β&&γ>f1&&γ>f2;
  lines:={{c,(β-δ-c*β+c*δ)/(β-δ+c*(α-2*β+δ))},{c,1}}/;γ==β&&γ<f1&&γ>f2;
  points:={{c,1}}/;f1>γ&&f1>f2&&f1>β&&γ!=β&&γ!=f2;
  points:={{c,1}}/;f1>γ&&f1>f2&&f1>β&&γ==β;
  points:={{c,1}}/;f1>γ&&f1>f2&&f1>β&&γ==f2;
  lines:={{c,0},{c,(β-δ-c*β+c*δ)/(β-δ+c*(α-2*β+δ))}}/;γ==β&&γ>f1&&γ<f2;
  points:={{c,0}}/;f2>γ&&f2>f1&&f2>β&&γ!=β&&γ!=f1;
  points:={{c,0}}/;f2>γ&&f2>f1&&f2>β&&γ==β;
  points:={{c,0}}/;f2>γ&&f2>f1&&f2>β&&γ==f1;
  lines:={{0,1},{c,1}}/;γ==β&&β==f1&&f1>f2;
  points:={{1,0}}/;γ==β&&β==f1&&f1>f2;
  lines:={{0,1},{c,1},{c,0}}/;γ==β&&β==f1&&f1<f2;
  points:={{0,1}}/;γ==β&&β==f2&&f2>f1;
  lines:={{c,0},{1,0}}/;γ==β&&β==f2&&f2>f1;
  lines:={{c,1},{c,0},{1,0}}/;γ==β&&β==f2&&f2<f1;
  lines:={{0,1},{c,1},{c,0},{1,0}}/;γ==β&&β==f2&&f2==f1;
  points:={{1,0}}/;β>f1&&β>f2&&β>γ;
  points:={{1,0},{c,1}}/;β==f1&&β>f2&&β>γ;
  lines:={{c,0},{1,0}}/;β>f1&&β==f2&&β>γ;
  lines:={{c,1},{c,0}}/;β==f1&&β<f2&&γ<β;
  lines:={{c,1},{c,0},{1,0}}/;β==f2&&β<f1&&γ<β;
  lines:={{c,1},{c,0},{1,0}}/;β==f2&&f2==f1&&f1>γ;
  lines:={{c,1},{c,0}}/;f1>γ&&f1==f2&&f1>β
]
  If[s>0&&t<0&&(s==-t),
  If[δ>α&&δ>β,g2:=Show[sq,QA,PQ,AR]];
  If[β>α&&β>δ,g2:=Show[sq,QA,PQ,QR]];
  If[α>β&&α>δ,g2:=Show[sq,QA,PQ,PR]];
  If[δ==α&&δ>β,g2:=Show[sq,QA,PQ,AR,PR]];
  If[δ==β&&δ>α,g2:=Show[sq,PQ,QAR]];
  If[(δ==α&&α==β)||(δ==β&&δ<α),g2:=Show[sq,QAR,PQR]];
  If[δ==α&&δ<β,g2:=Show[sq,QA,PQR,A]];
  If[α==β&&β>δ,g2:=Show[sq,QA,PQR]];

  If[α>δ&&α>β&&δ>β,g2:=Show[sq,QA,PQ,A,Graphics[{k,RGBColor[.49,0,0],Line
  [{{1,(δ-β)/(α-β)},{1,1}}]}]]];

  If[β>δ&&β>α&&δ>α,g2:=Show[sq,QA,PQ,A,Graphics[{k,RGBColor[.49,0,0],Line
  [{{1,0},{1,(δ-β)/(α-β)}}]}]]];

  points:={{0,0}}/;δ>α&&δ>β;
  points:={{1,0}}/;β>α&&β>δ;
```

```
  points:={{1,1}}/;α>β&&α>δ;
  points:={{0,0},{1,1}}/;δ==α&&δ>β;
  lines:={{0,0},{1,0}}/;δ==β&&δ>α;
  lines:={{0,0},{1,0},{1,1}}/;δ==α&&α==β;
  lines:={{0,0},{1,0},{1,1}}/; δ==β&&δ<α;
  lines:={{1,0},{1,1}}/;δ==α&&δ<β;
  lines:={{1,0},{1,1}}/;α==β&&β>δ;
  lines:={{1,(δ-β)/(α-β)},{1,1}}/;α>δ&&α>β&&δ>β;
  lines:={{1,0},{1,(δ-β)/(α-β)}}/; β>δ&&β>α&&δ>α
]
If[s<0&&t>0&&(-s==t),
  If[γ>α&&γ>β,g2:=Show[sq,BP,PQ,BR]];
  If[α>β&&α>γ,g2:=Show[sq,BP,PQ,PR]];
  If[β>α&&β>γ,g2:=Show[sq,BP,PQ,QR]];
  If[γ==α&&γ>β,g2:=Show[sq,PQ,BPR]];
  If[γ==β&&γ>α,g2:=Show[sq,BP,PQ,BR,QR]];
  If[(γ==α&&α==β)||(γ==α&&γ<β),g2:=Show[sq,BPR,PQR]];
  If[γ==β&&γ<α,g2:=Show[sq,BP,PQR,B]];
  If[α==β&&β>γ,g2:=Show[sq,BP,PQR]];

  If[α>γ&&α>β&&γ>β,g2:=Show[sq,BP,PQ,B,Graphics[{k,RGBColor[.49,0,0],Line
  [{{1,(γ-β)/(α-β)},{1,1}}]}]]];

  If[β>γ&&β>α&&γ>α,g2:=Show[sq,BP,PQ,B,Graphics[{k,RGBColor[.49,0,0],Line
  [{{1,0},{1,(γ-β)/(α-β)}}]}]]];

  points:={{0,1}}/;γ>α&&γ>β;
  points:={{1,1}}/;α>β&&α>γ;
  points:={{1,0}}/;β>α&&β>γ;
  lines:={{0,1},{1,1}}/;γ==α&&γ>β;
  points:={{0,1},{1,0}}/;γ==β&&γ>α;
  lines:={{1,1},{1,0},{0,1}}/;γ==α&&α==β;
  lines:={{1,1},{1,0},{0,1}}/;γ==α&&γ<β;
  lines:={{1,1},{1,0}}/;γ==β&&γ<α;
  lines:={{1,1},{1,0}}/;α==β&&β>γ;
  lines:={{1,(γ-β)/(α-β)},{1,1}}/; α>γ&&α>β&&γ>β;
  lines:={{1,0},{1,(γ-β)/(α-β)}}/;β>γ&&β>α&&γ>α
]
If[s>0&&t==0,
  If[α>δ&&α>γ,g2:=Show[sq,AB,BP,PR]];
  If[γ>δ&&γ>α,g2:=Show[sq,AB,BP,BR]];
  If[δ>γ&&δ>α,g2:=Show[sq,AB,BP,AR]];
  If[α==γ&&α>δ,g2:=Show[sq,AB,BPR]];
  If[α==δ&&α>γ,g2:=Show[sq,AB,BP,AR,PR]];
  If[(α==γ&&γ==δ)||(α==γ&&α<δ),g2:=Show[sq,ABR,BPR]];
  If[α==δ&&α<γ,g2:=Show[sq,BP,ABR,P]];
  If[γ==δ&&δ>α,g2:=Show[sq,BP,ABR]];

  If[γ>α&&γ>δ&&α>δ,g2:=Show[sq,AB,BP,P,Graphics[{k,RGBColor[.49,0,0],Line
  {{0,(α-δ)/(γ-δ)},{0,1}}]}]]]];

  If[δ>α&&δ>γ&&α>γ,g2:=Show[sq,AB,BP,P,Graphics[{k,RGBColor[.49,0,0],Line
  [{{0,0},{0,(α-δ)/(γ-δ)}}]}]]]];

  points:={{1,1}}/;α>δ&&α>γ;
  points:={{0,1}}/;γ>δ&&γ>α;
```

```
  points:={{0,0}}/;δ>γ&&δ>α;
  lines:={{0,1},{1,1}}/;α==γ&&α>δ;
  points:={{0,0},{1,1}}/;α==δ&&α>γ;
  lines:={{0,0},{0,1},{1,1}}/;α==γ&&γ==δ;
  lines:={{0,0},{0,1},{1,1}}/;α==γ&&α<δ;
  lines:={{0,0},{0,1}}/;α==δ&&α<γ;
  lines:={{0,0},{0,1}}/;γ==δ&&δ>α;
  lines:={{0,(α-δ)/(γ-δ)},{0,1}}/;γ>α&&γ>δ&&α>δ;
  lines:={{0,0},{0,(α-δ)/(γ-δ)}}/;δ>α&&δ>γ&&α>γ
]
If[s<0&&t==0,
  If[β>δ&&β>γ,g2:=Show[sq,AB,QA,QR]];
  If[δ>β&&δ>γ,g2:=Show[sq,AB,QA,AR]];
  If[γ>δ&&γ>β,g2:=Show[sq,AB,QA,BR]];
  If[β==δ&&β>γ,g2:=Show[sq,AB,QAR]];
  If[β==γ&&β>δ,g2:=Show[sq,AB,QA,BR,QR]];
  If[(β==γ&&γ==δ)|| (β==δ&&β<γ),g2:=Show[sq,ABR,QAR]];
  If[β==γ&&β<δ,g2:=Show[sq,QA,ABR,Q]];
  If[γ==δ&&δ>β,g2:=Show[sq,QA,ABR]];

  If[γ>β&&γ>δ&&β>δ,g2:=Show[sq,AB,QA,Q,Graphics[{k,RGBColor[.49,0,0],Line
  [{{0,(β-δ)/(γ-δ)},{0,1}}]}]]];

  If[δ>β&&δ>γ&&β>γ,g2:=Show[sq,AB,QA,Q,Graphics[{k,RGBColor[.49,0,0],Line
  [{{0,0},{0,(β-δ)/(γ-δ)}}]}]]];

  points:={{1,0}}/;β>δ&&β>γ;
  points:={{0,0}}/;δ>β&&δ>γ;
  points:={{0,1}}/;γ>δ&&γ>β;
  lines:={{0,0},{1,0}}/;β==δ&&β>γ;
  points:={{0,1},{1,0}}/;β==γ&&β>δ;
  lines:={{0,1},{0,0},{1,0}}/; β==γ&&γ==δ;
  lines:={{0,1},{0,0},{1,0}}/;β==δ&&β<γ;
  lines:={{0,1},{0,0}}/;β==γ&&β<δ;
  lines:={{0,1},{0,0}}/;γ==δ&&δ>β;
  lines:={{0,(β-δ)/(γ-δ)},{0,1}}/;γ>β&&γ>δ&&β>δ;
  lines:={{0,0},{0,(β-δ)/(γ-δ)}}/;δ>β&&δ>γ&&β>γ
]
If[(s>0&&t>0)||(s==0&&t>0)||(s<0&&t>0&&(-s<t)),
  If[α>γ,g2:=Show[sq,BP,PR]];
  If[α<γ,g2:=Show[sq,BP,BR]];
  If[α==γ,g2:=Show[sq,BPR]];

  points:={{1,1}}/;α>γ;
  points:={{0,1}}/;α<γ;
  lines:={{0,1},{1,1}}/;α==γ
]
If[(s<0&&t<0)||(s>0&&t<0&&(s<-t))||(s==0&&t<0),
  If[β>δ,g2:=Show[sq,QA,QR]];
  If[β<δ,g2:=Show[sq,QA,AR]];
  If[β==δ,g2:=Show[sq,QAR]];

  points:={{1,0}}/;β>δ;
  points:={{0,0}}/;β<δ;
  lines:={{0,0},{1,0}}/;β==δ
]
```

```
If[s==0&&t==0,
  If[α>β&&α>δ&&α>γ,g2:=Show[sr,PR]];
  If[γ>β&&γ>δ&&γ>α,g2:=Show[sr,BR]];
  If[β>α&&β>δ&&β>γ,g2:=Show[sr,QR]];
  If[δ>β&&δ>α&&δ>γ,g2:=Show[sr,AR]];
  If[α==β&&α>δ&&α>γ,g2:=Show[sr,PQR]];
  If[γ==δ&&γ>α&&γ>β,g2:=Show[sr,ABR]];
  If[α==γ&&α>δ&&α>β,g2:=Show[sr,BPR]];
  If[δ==β&&β>α&&β>γ,g2:=Show[sr,QAR]];
  If[δ==α&&α>β&&α>γ,g2:=Show[sr,AR,PR]];
  If[γ==β&&β>α&&β>δ,g2:=Show[sr,BR,QR]];
  If[α==β&&β==δ&&α>γ,g2:=Show[sr,QAR,PQR]];
  If[α==β&&β==γ&&α>δ,g2:=Show[sr,BPR,PQR]];
  If[α==δ&&δ==γ&&α>β,g2:=Show[sr,ABR,BPR]];
  If[γ==β&&β==δ&&γ>α,g2:=Show[sr,QAR,ABR]];
  If[α==β&&β==δ&&δ==γ,g2:=Graphics[{Thick,RGBColor[.49,0,0],Polygon[{{0,0},
  {0,1},{1,1},{1,0}}]}]]];
  points:={{1,1}}/;α>β&&α>δ&&α>γ;
  points:={{0,1}}/;γ>β&&γ>δ&&γ>α;
  points:={{1,0}}/;β>α&&β>δ&&β>γ;
  points:={{0,0}}/;δ>β&&δ>α&&δ>γ;
  lines:={{1,1},{1,0}}/;α==β&&α>δ&&α>γ;
  lines:={{0,0},{0,1}}/;γ==δ&&γ>α&&γ>β;
  lines:={{0,1},{1,1}}/;α==γ&&α>δ&&α>β;
  lines:={{0,0},{1,0}}/;δ==β&&β>α&&β>γ;
  points:={{0,0},{1,1}}/;δ==α&&α>β&&α>γ;
  points:={{0,1},{1,0}}/;γ==β&&β>α&&β>δ;
  lines:={{0,0},{1,0},{1,1}}/;α==β&&β==δ&&α>γ;
  lines:={{0,1},{1,1},{1,0}}/;α==β&&β==γ&&α>δ;
  lines:={{0,0},{0,1},{1,1}}/;α==δ&&δ==γ&&α>β;
  lines:={{0,1},{0,0},{1,0}}/;γ==β&&β==δ&&γ>α;
  lines:={{0,0},{0,1},{1,1},{1,0},{0,0}}/;α==β&&β==δ&&δ==γ
]
SE:=Graphics[{{PointSize[Large],RGBColor[.49,0,0],Point[points]},
{PointSize[Large],RGBColor[.49,0,0],Point[lines]},{RGBColor[.49,0,0],
Thickness[0.013], Line[lines]}},ImageSize→{400,400}];
Grid[{{Show[g2,Axes→True,PlotRange→All,ImageSize→{400,400}]},
{" "},{Text@Style["vertices of the set of Stackelberg
Equilibria",Bold]},{Text[pointsUlines]}},ItemSize→{Automatic,{10,1,1,3}},
Alignment→{Center,Top}]
]
Manipulate[
  GR2[{{a11,a12},{a21,a22}},{{b11,b12},{b21,b22}}],
    Style["elements of payoff matrix A",Bold],
    {{a11,-5,"a11"},-10,10,1,Appearance→ "Labeled",ImageSize→Tiny},
    {{a12,-3,"a12"},-10,10,1,Appearance→ "Labeled",ImageSize→Tiny},
    {{a21,-1,"a21"},-10,10,1,Appearance→ "Labeled",ImageSize→Tiny},
    {{a22,-4,"a22"},-10,10,1,Appearance→ "Labeled",ImageSize→Tiny},
    Delimiter,Style["elements of payoff matrix B",Bold],
    {{b11,-5,"b11"},-10,10,1,Appearance→ "Labeled",ImageSize→Tiny},
    {{b12,-3,"b12"},-10,10,1,Appearance→ "Labeled",ImageSize→Tiny},
    {{b21,-1,"b21"},-10,10,1,Appearance→ "Labeled",ImageSize→Tiny},
```

```
{{b22,-4,"b22"},-10,10,1,Appearance→ "Labeled",ImageSize→Tiny},
Delimiter,
Style["payoff matrices A and B",Bold],
Dynamic[TableForm[{{ToString[a11]<>", "<>ToString[b11],
ToString[a12]<>", "<>ToString[b12]}, {ToString[a21]<>",
"<>ToString[b21],ToString[a22]<>",
"<>ToString[b22]}},TableHeadings→{{"1","2"},{" 1",
" 2"}},TableSpacing→{2,2}]],SaveDefinitions→True]
```

Execuția programului poate fi vizualizată direct pe adresa indicată în [6].

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Prezentat la 16.02.2010