STUDIA UNIVERSITATIS

*Revistă științifică a Universității de Stat din Moldova, 2008, nr.3(13)*

# NASH EQUILIBRIA IN THE TWO PLAYERS NONCOOPERATIVE INFORMATIONAL EXTENDED GAMES

***Ludmila NOVAC***

*Catedra Informatică şi Optimizare Discretă*

Articolul conține descrierea jocurilor noncooperatiste informațional extinse, adică astfel de jocuri în care participanții cunosc strategiile alese ale adversarilor săi. Pentru aceste tipuri de jocuri informațional extinse se analizează existența situațiilor Nash de echilibru. Este formulată şi demonstrată teorema de existență a situațiilor Nash de echilibru pentru jocurile noncooperatiste informațional extinse de două persoane. La demonstrarea acestei teoreme este utilizată teorema Kakutani despre punctele fixe pentru aplicațiile multivoce.

## 1. Preliminary facts

### *1.1. Fixed points and contraction mappings*

Consider the function $f: X \to X$. An element $x \in X$ is called fixed point of $f$ if $f(x)=x$.
The fixed points of the function $f$ are the intersection points of the graph of $f$ with the product $X \times X$.

*Properties of fixed points*

**1.** If there are two functions $f$ and $g$ from $X$ into $Y$, then the point $x^* \in X$ for which $f(x^*)=g(x^*)$, is called point of coincidence for the functions $f$ and $g$.

**2.** Sometimes it is convenient to use the cyclic points of the function $f$ together with the fixed points, especially in the case when fixed points do not exist. Cyclic points are called the points which are images of the iterative function $f^n$, where $n$ is a natural number. These are cyclic points of the $n^{th}$ order. Often such points do not exist and in these cases we can use limit cycles. Also we can speak about the invariant sets, i.e. subsets $Y \subset X$, for which $f(Y)=Y$. In such cases the minimal invariant subsets are very important.

The notation $F: X \to 2^Y$ will denote a point-to-set mapping, were $2^Y$ denotes the set of all subsets of $Y$. A fixed point of the point-to-set mapping $F: X \to 2^Y$ is called a point $x^* \in X$, such that $x^* \in F(x^*)$. The graph for the application $F$ is called the set $gr(F) = \{(x,y) \in X \times Y \mid x \in X, y \in F(x)\}$ . This set can contain some points or can be the empty set.

### *1.2. The Kakutani fixed point theorem*

The existence of the fixed points is considered an important problem. The existence (and other properties) of the fixed point for the function $f: X \to X$ depends on the properties of $f$ and on the properties of the space $X$. Often it is considered that $f$ is a continuous function.

**Definition 1.** The function $f$ of the metric space into itself is called contraction mapping if there exists constant $K<1$, such that for each two points $x$ and $y$ the inequality $\rho(f(x),f(y)) \le K\rho(x,y)$ holds.

There are some important properties for the fixed points.

**Proposition 1.** If f is a contraction mapping, then there exists not more than a single fixed point (see [2], [3]).

**Theorem 1.** (Principle of the contraction mapping). Consider that $f$ is a contraction mapping of the complete metric space $X$ into itself. Then for each point $x \in X$ the sequence $x, f(x), f^2(x)=f(f(x)), f^3(x)$,... converges to a fixed point. So $f$ has a single fixed point ([2], [3]).

The points $x, f(x), f^2(x)$,... are called consequent approximations of the fixed point.

In the case of the contraction mapping we can consider as a start element every element $x$ and the consecutive approximations converge to the fixed point.

**The Kakutani fixed point theorem** is a fixed-point theorem for point-to-set mapping. It provides sufficient conditions for a point-to-set mapping defined on a convex, compact subset of a Euclidean space to have a fixed point, i.e. a point which is mapped to a set containing it. The Kakutani fixed point theorem is a generalization of Brouwer fixed point theorem. The Brouwer fixed point theorem is a fundamental result in topology which proves the existence of fixed points for continuous functions defined on compact, convex subsets of Euclidean spaces. Kakutani theorem extends this to point-to-set mapping.

The theorem was developed by Shizuo Kakutani in 1941 and was famously used by John Nash in his description of Nash equilibrium. It has subsequently found widespread application in game theory and economics. Many problems in economy appear as problems of maximization and usually the solution of such problems is many-valued.

Before giving this theorem we need to recall some definitions.

**Definition 2.** Consider topological spaces $X$ and $Y$. A point-to-set mapping $F: X \rightrightarrows 2^Y$ is said to be closed if the graph of $F$ is closed as a subset into the product of the spaces $X\times Y$.

That is if the sequence of points $(x_n,y_n)$ from $gr(F)$ converges to a point $(x,y)\in X\times Y$, then the limit point $(x,y)\in gr(F)$ [3].

**Theorem** (**Kakutani**). (1941). Let $X$ be a Banach space and $K$ a non-empty, compact and convex subset of $X$. Let $F: K \rightrightarrows 2^K$ be a point-to-set mapping on $K$ with a closed graph and the property that the set $F(x)$ is non-empty and convex for all $x\in K$. Then $F$ has a fixed point.

For proof see Berge ([1], p.74-76), [2].

Before giving the applications of the fixed points in the game theory we will recall some other important theorems.

Let $C(K)$ be the space of all continuous functions defined on the compactum $K$.

**Theorem** (**Arzelà-Ascoli**). (Compactness criterion). A set of continuous functions $E\subseteq C(k)$ is compact if and only if the set $E$ is uniformly bounded: ($|x(t)|\le N$, $\forall t\in K$, for $\forall x\in E$) and the functions from the set $E$ are equicontinuous (i.e. for $\forall\varepsilon$, $\exists\delta$ so that if $\rho(t_1,t_2)<\delta$ then $|x(t_1)-x(t_2)|<\varepsilon$ for $\forall x\in E$).

**Theorem** (**Tihonov**). A product of a family of compact topological spaces $X=\prod_{\alpha\in A} X_\alpha$ is compact.

**Lemma [4].** 1) If $X$ and $Y$ are two compactums with the same metric and $f:X\to Y$ is a continuous function, then the set $Arg\max_{x\in X} f(x)=\left\{x\in X \mid f(x)=\max_{z\in X} f(z)\right\}$ is compact too.

2) If $X$ and $Y$ are two compactums with the same metric, and $K(x,y)$ is a continuous function on $X\times Y$, then $\varphi(y)=\max_{x\in X} K(x,y)$ and $\psi(x)=\min_{y\in Y} K(x,y)$ are continuous functions on $Y$ and $X$ respectively.

## 2. Strategic form games and Nash equilibriums

In this part we will analyse games in which the players choose their actions simultaneously (without the knowledge of other player choices). The game will assume that players payoff functions are common knowledge.

**Definition 3.** A strategic form of the game consists of: a finite set of players $I=\{1,2,...,n\}$, action spaces (set of strategies) of players, denoted by $X_i, i\in I$; and payoff functions of players $H_i: X\to R, i\in I$, where $X=X_1\times...\times X_n$. We refer to such a game as the tuple $\langle I,(X_i)_{i\in I},(H_i)_{i\in I}\rangle$ denoted by $\Gamma$.

An outcome is an action profile $(x_1, x_2, ..., x_n)$, and the outcome space is $X=\times_{i\in I} X_i$. The game is common knowledge among the players.

One of the most common interpretations of Nash equilibrium (introduced by John Nash in 1950) is that it is a steady state in the sense that no rational player has an incentive to unilaterally deviate from it.

Let $x_{-i}\equiv(x_1, x_2,..., x_{i-1}, x_{i+1},..., x_n)$ and $(x_{-i},y_i)\equiv(x_1, x_2,...,x_{i-1}, y_i, x_{i+1},..., x_n)$.

**Definition 4.** A Nash equilibrium of the game $\Gamma$ is an action profile $x^*\in X$ such that for every $i\in I$ the relations $H_i(x^*)\ge H_i(x^*_{-i},x_i)$ hold for all $x_i\in X_i$.

Another and sometimes a more convenient way of defining Nash equilibrium is via best response correspondences $Br_i: \times X_j \rightrightarrows Xi$ such that

$$Br_i(x_{-i})=\{x_i\in X_i : H_i(x)\ge H_i(x_{-i},x_i')\, for \forall x_i'\in X_i\}. \quad (*)$$

**Definition 5.** A Nash equilibrium is an action profile $x^*$ such that $x_i^* \in Br_i(x_{-i}^*)$ for all $i \in I$.

If the sets $X_i$ are compacts and the functions $H_i$ are continuous, then the best response set (*) for the player *i* can be represented by:

$$Br_i(x_{-i}) = Arg \max_{x_i \in X_i} H_i(x_{-i}, x_i).$$

Given a strategic form of the game $\Gamma \equiv \langle I, (X_i)_{i\in I}, (H_i)_{i\in I} \rangle$, the set of Nash equilibria is denoted by $NE(\Gamma)$.

Using the best response sets of the players we consider the point-to-set mapping $Br: \times X_i \rightrightarrows 2^X$ such that $Br = (Br_1, Br_2, \ldots, Br_n)$.

Then we can easily prove that $x^* \in NE(\Gamma)$ if and only if $x^*$ is a fixed point of the set-valued mapping *Br*, i.e. $x^* \in Br(x^*)$.

**3. Applications of the fixed point theorem for the two players informational extended games**

***Description of the game:***

Consider a static game with two players and consider that the second player knows the chosen strategy of the first player.

The game is realised as follows: the players choose his strategies simultaneously, after that each of them determines his payoff and the game is over.

Let us denote by $C(X,Y)$ the space of all continuous functions from *X* into *Y*, were *X* and *Y* are compactums.

Let us define this game in the normal form by: ${}_2\Gamma = \langle X, \overline{Y}, H_1, H_2 \rangle$, where $\overline{Y}$ represents the set of strategies for the second player and is defined by $\overline{Y} = \{\varphi: X \rightarrow Y\}$, the functions $\varphi \in C(X,Y)$ are continuous on the compactum *X*, the payoff functions for the players are defined on the product of the sets of strategies: $H_i : X \times \overline{Y} \rightarrow R$, $(i=1,2)$.

Next we will prove the following

**Theorem.** Let us consider that for the game ${}_2\Gamma$ the next conditions hold:

1) *X* and *Y* are non-empty compact and convex sets of Banach space,

2) the set of functions $\overline{Y} \subset C(X,Y)$ is uniformly bounded and the functions from the set $\overline{Y}$ are equicontinuous;

3) the real-valued functions $H_1(x, \varphi(x))$, $H_2(x, \varphi(x))$ are continuous on the compact $X \times \overline{Y}$ and concave on *X*, (on $\overline{Y}$, respectively).

Then $NE({}_2\Gamma) \neq \varnothing$.

**Proof**. Let $S = X \times \overline{Y}$ be the outcome space. According to Arzelà-Ascoli theorem the set $\overline{Y}$ is compact, and according to Tihonov theorem the outcome space S is compact too.

We define the point-to-set mapping $B: S \rightrightarrows 2^S$, such that $B(s) = (B_1(\varphi), B_2(x))$, where $B_1(\varphi)$, $B_2(x)$ represent the best response sets for the first and second player, respectively.

Because *X* and $\overline{Y}$ are compacts and $H_1$, $H_2$ are continuous functions, then according to Weierstrass theorem we can write:

$$B_1(\varphi) = Arg \max_{x \in X} H_1(x, \varphi(x)),$$

$$B_2(x) = Arg \max_{\varphi \in \overline{Y}} H_2(x, \varphi(x)),$$

i.e.:

$$B_1(\varphi) = \{x \in X : H_1(x, \varphi(x)) = \max_{z \in X} H_1(z, \varphi(z))\}$$

$$B_2(x) = \{\varphi \in \overline{Y} : H_2(x, \varphi(x)) = \max_{\psi \in \overline{Y}} H_2(x, \psi(x))\}$$

In order to use the Kakutani theorem we need to prove that:

1) $S = X \times \overline{Y} \neq \varnothing$ is non-empty convex compact set;

2) for the point-to-set mapping $B: S \rightrightarrows 2^S$ the next conditions hold:

a) for $\forall x \in X$, $\forall \varphi \in \overline{Y}$ the set $B(x,\varphi)\neq\varnothing$ is a convex subset of $S$;

b) the point-to-set mapping $B$ is closed.

Firstly we will prove that $S$ is convex and compact.

The set $\overline{Y}$ is convex if: for $\forall \varphi_1, \varphi_2 \in \overline{Y}$, and $\lambda \in [0,1]$ the function $\lambda\varphi_1 + (1-\lambda)\varphi_2$ is bounded by the same constant $N$ (see Arzelà-Ascoli theorem) and the function $\lambda\varphi_1 + (1-\lambda)\varphi_2$ is equicontinuous.

It is easy to prove that the function $\lambda\varphi_1 + (1-\lambda)\varphi_2$ is bounded by the same constant $N$: $|\lambda\varphi_1(x) + (1-\lambda)\varphi_2(x)| \le \lambda|\varphi_1(x)| + (1-\lambda)|\varphi_2(x)| \le \lambda N+(1-\lambda)N = N$ for $\forall \varphi_1, \varphi_2 \in \overline{Y}$, and $\lambda \in [0,1]$.

Evidently the function $\lambda\varphi_1 + (1-\lambda)\varphi_2$ is equicontinuous. So the set $\overline{Y}$ is convex. Then the set $S$ is convex and compact too.

Next we need to prove that for the point-to-set mapping $B: S \rightrightarrows 2^S$ the conditions a) and b) hold.

Firstly we will prove the condition a). For $\forall x \in X$ and $\forall \varphi \in \overline{Y}$ the set $B(x,\varphi)$ is non-empty, this follows from the Weierstrass theorem, because $B_1(\varphi)$ and $B_2(x)$ are non-empty sets.

Next we need to prove that the set $B(x,\varphi)$ is convex for $\forall x \in X$, $\forall \varphi \in \overline{Y}$.

Firstly we will prove that $B_1(\varphi)$ is a convex set. Let us suppose that there are two elements $x_1, x_2 \in B_1(\varphi)$ and $0\le\lambda\le1$, then because $H_1$ is a concave function it follows that:

$$H_1(\lambda x_1+(1-\lambda)x_2,\varphi) \ge \lambda H_1(x_1,\varphi)+(1-\lambda)H_1(x_2,\varphi) =$$

$$= \lambda \max_{z\in X} H_1(z,\varphi(z))+(1-\lambda) \max_{z\in X} H_1(z,\varphi(z)) = \max_{z\in X} H_1(z,\varphi(z)).$$

On the other hand, since $X$ is a convex compactum, then $\lambda x_1 + (1-\lambda)x_2 \in X$, so it follows that $H_1(\lambda x_1+(1-\lambda)x_2,\varphi) \le \max_{z\in X} H_1(z,\varphi(z))$. Then it results that:

$H_1(\lambda x_1 + (1-\lambda)x_2,\varphi) = \max_{z\in X} H_1(z,\varphi(z))$; thus $\lambda x_1 + (1-\lambda)x_2 \in B_1(\varphi)$, which implies that $B_1(\varphi)$ is a convex set.

Next we will prove that the set $B_2(x)$ is convex too.

The function $H_2(x,\varphi(x))$ is concave on the compact set $\overline{Y} \subset C(X,Y)$, then by definition for $\forall \lambda \in [0,1]$, and $\forall \varphi_1, \varphi_2 \in \overline{Y}$ the relation $H_2(x,\lambda\varphi_1+(1-\lambda)\varphi_2)\ge\lambda H_2(x,\varphi_1)+(1-\lambda)H_2(x,\varphi_2)$ holds.

To prove that the set $B_2(x)$ is convex we need to prove that $\lambda\varphi_1 + (1-\lambda)\varphi_2 \in \overline{Y}$.

Consider the functions $\varphi_1, \varphi_2 \in \overline{Y}$ which are bounded and equicontinuous.

Evidently, the function $\lambda\varphi_1+(1-\lambda)\varphi_2$ is equicontinuous, so it follows that $\lambda\varphi_1+(1-\lambda)\varphi_2\in B_2(x)$, thus $B_2(x)$ is a convex set.

From what was proved it follows that for $\forall x \in X$ and $\forall \varphi \in \overline{Y}$ we will have a convex subset $B(s)=(B_1(\varphi),B_2(x))\neq\varnothing$ from $S=X\times\overline{Y}$.

Next we will prove the condition b). We need to prove that the point-to-set mapping $B$ is closed.

Analyse the point-to-set mapping $B: S \to 2^S$ which maps the point $(x,\varphi)\in S$ to the set $B_1(\varphi)\times B_2(x) \subset S$.

The point-to-set mapping $B$ is closed if its graph is a closed set [4]. Since $B_1(\varphi)$ is a subset from the compactum $X$, and $B_2(x)$ is a subset from the compactum $\overline{Y}$, then $grB_1(\varphi)$ and $grB_2(x)$ are compact sets. Here the graphs for the sets $B_1(\varphi)$ and $B_2(x)$ are defined by:

$$grB_1 = \left\{(x,\varphi) \mid x \in Arg \max_{z\in X} H_1(z,\varphi(z)), \varphi \in \overline{Y}\right\},$$

$$grB_2 = \left\{(x,\varphi) \mid x \in X, \varphi \in Arg \max_{\psi\in \overline{Y}} H_2(x,\psi(x))\right\}.$$

We will prove that the sets $B_1(\varphi)$ and $B_2(x)$ are closed.

The set $B_1(\varphi)$ can be rewritten as follows:

$$B_1(\varphi)=\left\{x\in X : H_1(x,\varphi(x)) - \max_{z\in X} H_1(z,\varphi(z)) = 0\right\}.$$

Because the set $X$ is compact and the function $H_1$ is continuous on $X$, then the function $H_1(x,\varphi(x))-\max_{z\in X}H_1(z,\varphi(z))$ is continuous on $X$ too. So for $\forall\varphi$ the set $B_1(\varphi)$ is closed (and compact). In a similar manner we will prove that the set $B_2(x)$ is closed: since $\overline{Y}$ is compact and the function $H_2$ is continuous on $\overline{Y}$, then the function $H_2(x,\varphi(x))-\max_{\psi\in\overline{Y}}H_2(x,\psi(x))$ is continuous on $\overline{Y}$ too, thus it follows that for $\forall x\in X$ the set $B_2(x)\subset\overline{Y}$ is closed.

Then accordingly to the Tihonov theorem, because

$grB_1=\{(x,\varphi(x))\in X\times\overline{Y} \mid x\in B_1(\varphi),\varphi(x)\in\overline{Y}\}$ is a closed set and

$grB_2=\{(x,\varphi(x))\in X\times\overline{Y} \mid x\in X,\varphi(x)\in B_2(x)\}$ is a closed set, it follows that

$grB=\{(x,\varphi(x))\in S \mid x\in B_1(\varphi(x)),\varphi(x)\in B_2(x)\}$ is a closed set too.

Thus the point-to-set mapping $B$ is closed.

Therefore we can apply the Kakutani theorem.

Let $(x_0,\varphi_0)\in S=X\times\overline{Y}$ be the fixed point for the point-to-set mapping $B$, i.e. $(x_0,\varphi_0)\in B(x_0,\varphi_0)=B_1(\varphi_0)\times B_2(x_0)$, so the relations

$$H_1(x_0,\varphi_0)=\max_{x\in X}H_1(x,\varphi_0),$$
$$H_2(x_0,\varphi_0)=\max_{\varphi\in\overline{y}}H_2(x_0,\varphi)$$

hold, thus by definition of the Nash equilibrium it follows that $(x_0,\varphi_0)\in NE({}_2\Gamma)\neq\varnothing$.■

**Repherences:**

1. Berge C. Topological spaces. - New York: Macmillan, 1963, p.74-76.
2. Канторович Л.В., Акилов Г.П. Функциональный анализ. - Москва: Наука, 1984. - 752 с.
3. Danilov V.I. Lectures on fixed points, 1998 (Russ.), p.29 http://www.nes.ru/russian/research/pdf/1998/Danilov.pdf
4. Hâncu B. Probleme de optimizare pe multe nivele. Partea I. - Chişinău: CE USM, 2002. - 103 p.

*Prezentat la 20.06.2008*