THE SPECIAL METRICS OF THE ABSTRACT CUBIC COMPLEX

Sergiu CATARANCIUC
Catedra Informatică şi Optimizare Discretă

Se examinează complexul cubic abstract K^n ca un caz particular al G-complexului de relaţii multi-are [2]. Pentru complexul cubic abstract K^n se defineşte o funcţie specială ce posedă proprietăţile metricii.

Let $X = \{x_1, x_2, ..., x_r, ...\}$ be a countable set of elements. We form the infinite series of Cartesian products:

$$X = X^1, X^2, ..., X^n, ...$$

where

$$X^{m+1} = X^m \cdot X, \quad m \geq 1.$$

Any non-empty subset $R^m \subset X^m, \quad m \geq 1$, is called an *m*-ary relation of the elements from *X*. We mention that the 1-ary relation $R^1 \subset X^1$ represents a subset of elements from *X*. Thus, an *m*-ary R^m relation is a family of ordered successions of the following type $\left(x_{i_1}, x_{i_2}, ..., x_{i_m}\right)$, where $x_{i_1}, x_{i_2}, ..., x_{i_m}$ are elements of *X*. The elements of the *m*-ary relation R^m will be called *corteges*. In general, any cortege $\left(x_{i_1}, x_{i_2}, ..., x_{i_m}\right)$ may contain also repetitions of the elements from *X*. For a cortege $\left(x_{i_1}, x_{i_2}, ..., x_{i_m}\right) \in R^m$, any subcortege $\left(x_{j_1}, x_{j_2}, ..., x_{j_l}\right), 1 \leq l \leq m$, that preserves the order of elements in $\left(x_{i_1}, x_{i_2}, ..., x_{i_m}\right)$, is called *hereditary subcortege*.

Definition 1. *The finite family of relations* $\left\{R^1, R^2, ..., R^{n+1}\right\}$ *that satisfies the following conditions:*

I. $R^1 = X^1 = X$,

II. $R^{n+1} \neq \varnothing$,

III. *any hereditary subseries* $\left(x_{j_1}, x_{j_2}, ..., x_{j_l}\right)$, $1 \leq l \leq m \leq n+1$ from $\left(x_{i_1}, x_{i_2}, ..., x_{i_m}\right) \in R^m$ belongs to the *l-ary relation* R^l,

IV. *for an arbitrary cortege from* $R^m, 1 \leq m \leq n+1$, *the set of all corteges from the family* $\left\{R^{m+1}, ..., R^{n+1}\right\}$, *which have the given cortege as an hereditary subcortege, is a finite set,*

is called $\mathcal{G}$*–complex of multi-ary relations and it is denoted by*

$$\mathcal{R}^{n+1} = \left\{R^1, R^2, ..., R^{n+1}\right\}.$$

The fundamental definitions related to the examination of a $\mathcal{G}$–complex of multi-ary relations, along with its properties are described in [1, 2].

The studying of such objects as the $\mathcal{G}$–complex of multi-ary relations is of interest by the fact that they generalize a series of classical discrete structures, as graphs, hypergraphs, the abstract simplicial complexes, etc., as well as by the possibility of elaborating efficient methods for solving some important applicative problems.

A particular case of the $\mathcal{G}$–complex of multi-ary relations, which appears in a lot of applicative problems, is the abstract cubic complex. The notion of cubic complex has been denoted for the first time in [4]. Thus, in the Euclidian space E^{n+1}, it is defined the following complex of finite-dimensional unitary cubes:

$$\mathcal{K}^n=\left\{I_\lambda^m : \lambda\in\Lambda,\, 0\le m\le n\right\},$$

as well as the series of direct cubic homologies groups $H_0\left(\mathcal{K}^n,\mathbf{Z}\right), H_1\left(\mathcal{K}^n,\mathbf{Z}\right),\ \ldots, H_n\left(\mathcal{K}^n,\mathbf{Z}\right)$ of $\mathcal{K}^n$ over the integer field $\mathbf{Z}$, [3, 4].

The definition of $\mathcal{K}^n$ is quite simple:

1) each facet $I^k\subset I^m,\, 0\le k\le m\le n$ is an element of $\mathcal{K}^n$;
2) for each couple of cubes $I_{\lambda_1}^{m_1}, I_{\lambda_2}^{m_2}\in\mathcal{K}^n$, the $I_{\lambda_1}^{m_1}\bigcap I_{\lambda_2}^{m_2}$ product us null or represents an element from $\mathcal{K}^n$.

If the $\mathcal{K}^n$ complex is connected, then the $H_0\left(\mathcal{K}^n,\mathbf{Z}\right)$ group is isomorphic with the set of integer numbers **Z**:

$$H_0\left(\mathcal{K}^n,\mathbf{Z}\right)\cong\mathbf{Z}$$

We will consider that for $\mathcal{K}^n$ are hold the following relations:

$$H_0\left(\mathcal{K}^n,\mathbf{Z}\right)\cong\mathbf{Z}, H_1\left(\mathcal{K}^n,\mathbf{Z}\right)\cong H_2\left(\mathcal{K}^n,\mathbf{Z}\right)\cong\ \ldots\cong H_n\left(\mathcal{K}^n,\mathbf{Z}\right)\cong 0\,.$$

Under these conditions, the complex $\mathcal{K}^n$ is called *acyclic complex*.

Next, we will consider that $\mathcal{K}^n$ is a non-oriented finite abstract cubic complex, formed by *m*-dimensional cubes I^m, $0\le m\le n$, and that contains at least one *n*-dimensional cube I^n. In this case we say that the complex $\mathcal{K}^n$ is *n*-dimensional. Evidently, some of the *m*-dimensional cubes from $\mathcal{K}^n$ are subcubes of the cubes with a greater dimension in $\mathcal{K}^n$. Geometrically, in the linear space R^n over the real field $\mathbf{R}$, the *m*-dimensional cubes of which it is formed the $\mathcal{K}^n$ $\mathcal{G}$-complex can be interpreted as follows:

It is denoted by Q^m the family of all the *m*-dimensional cubes from $\mathcal{K}^n$:

$$Q^m=\left\{I^m : I^m\in\mathcal{K}^n\right\},\ 0\le m\le n.$$

Definition 2: *It is called a frontier of the cubic* $\mathcal{G}$*-compl*$\mathcal{K}^n$ *, the set of all* $(n-1)$*-dimensional cubes that belong to at most one n-dimensional cube, denoted by* $bd^{n-1}\mathcal{K}^n$.

So, the 2-dimensional frontier of an abstract 3-dimensional cube I^3 is formed by all the 2-dimensional facets of this cube, and topologically frames an abstract 2-dimensional sphere. If $\mathcal{K}^3$ is consisted of two 3-dimensional cubes with a single common vertex, then $bd^{n-1}\mathcal{K}^n$ topologically represents two abstract 2-dimensional spheres with a common point.

Let now I^n be an *n*-dimensional cube, and VI^n its vacuum (the vacuum definition can be found in [4]). We mention that by definition $VI^0=I^0$.

Definition 3: *The union of vacuums of all the abstract cubes in the* $\mathcal{G}$*-complex* $\mathcal{K}^n$ *, that do not belong to the frontier* $bd^{n-1}\mathcal{K}^n$ *is called the interior of the* $\mathcal{G}$*-complex* $\mathcal{K}^n$*, denoted by* $int^n\mathcal{K}^n$.

Let $Q=\bigcup_{m=0}^{n} Q^m$ be the family of all the cubes in $\mathcal{K}^n$. We denote by $Q_0\subset Q$ the family of all the *m*-dimensional cubes from $\mathcal{K}^n$ that don't belong to this ones' frontier. Thus:

$$\operatorname{int}\mathcal{K}^n=\bigcup_{I^m\in Q_0} VI^m\,.$$

We will examine forwards the abstract cubic $\mathcal{G}$-complex $\mathcal{K}^n$, connected and acyclic, with the following properties:

1) if $I^0 \in int^n \mathcal{K}^n$, then I^0 belongs to at least 2^n n-dimensional cubes;
2) if $I^0 \in bd^{n-1} \mathcal{K}^n$, and belongs to at least 2 n-dimensional I^n cubes from $\mathcal{K}^n$, then I^0 belongs to a number not less than $n+1$ of $(n-1)$-dimensional cubes.

Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ be the classes of parallel edges of the abstract cubic complex $\mathcal{K}^n$. We choose two arbitrary elements: $I_p^0, I_q^0 \in \mathcal{C}^0$. We mention that the $\mathcal{C}^0$ set can be considered as the set of vertices of the graph from the 1-dimensional skeleton of $skl\ \mathcal{K}^n$.

It will be denoted by $\mathcal{L}^1$ the set of all the linear chains of the $\mathcal{K}^1$ subcomplex, which is in fact the 1-dimensional skeleton of the abstract cubic complex $\mathcal{K}^n$. We define the function $d: \mathcal{L}^1 \to \mathbb{R}^+$ on $\mathcal{L}^1$, so that, if the chain $L^1 \in \mathcal{L}^1$, then

$$d\left(L^1\right) = \sum_{k=1}^{m} \varepsilon_k d_k , \tag{1}$$

Where $d_k \in \mathbb{R}^+$ represents the weight of the parallel edges class, $\mathcal{C}_k$, $k = \overline{1,m}$, and

$$\varepsilon_k = \begin{cases} 0, \text{ if the } L^1 \text{ chain intersects the } C_k \text{ class an even number of times} \\ 1, \text{ if the } L^1 \text{ chain intersects the } C_k \text{ class an uneven number of times.} \end{cases}$$

(in the case when L^1 doesn't contain edges from $\mathcal{C}_k$ it will be considered that the number of intersections with this particular class is even). If $I_p^0, I_q^0 \in \mathcal{C}^0$ are extremities of the $L^1 = L^1\left(I_p^0,\ I_q^0\right)$ then it will be used the following notation:

$$d_{L^1}\left(I_p^0,\ I_q^0\right) = d\left(L^1\right) = \sum_{k=1}^{m} \varepsilon_k d_k .$$

The $d\left(L^1\right)$ number will be called the length of the $L^1\left(I_p^0,\ I_q^0\right)$ chain.

Theorem 1: *If* $L_1^1\left(I_p^0,\ I_q^0\right), L_2^1\left(I_p^0,\ I_q^0\right) \in \mathcal{L}^1$ *are two distinct linear chains that connect the vertices* $I_p^0, I_q^0 \in \mathcal{C}^0$*, then:*

$$d_{L_1^1}\left(I_p^0,\ I_q^0\right) = d_{L_2^1}\left(I_p^0,\ I_q^0\right).$$

Proof: Let $L_1^1\left(I_p^0,\ I_q^0\right)$ and $L_2^1\left(I_p^0,\ I_q^0\right)$ be two distinct chains which connect the vertices $I_p^0, I_q^0 \in \mathcal{C}^0$. We form the following union: $L' = L_1^1\left(I_p^0,\ I_q^0\right) \cup L_2^1\left(I_p^0,\ I_q^0\right)$ which is a 1-dimensional cycle and which, accordingly, intersects each class of parallel edges $\mathcal{C}_k, k = \overline{1,m}$, an even number of times, given that the abstract cubic complex $\mathcal{K}^n$, examined above, is acyclic. This means that the number of intersections between the chain $L_1^1\left(I_p^0,\ I_q^0\right)$ with the class of parallel edges $\mathcal{C}_k, k = \overline{1,m}$ and the number of intersections between the chain $L_2^1\left(I_p^0,\ I_q^0\right)$ with the class $\mathcal{C}_k, k = \overline{1,m}$ are both of the same parity. Thus, if we denote by

$$d_{L_1^1}\left(I_p^0,\ I_q^0\right) = d\left(L_1^1\right) = \sum_{k=1}^{m} \varepsilon_k^1 d_k$$

$$d_{L_2^1}\left(I_p^0,\ I_q^0\right) = d\left(L_2^1\right) = \sum_{k=1}^{m} \varepsilon_k^2 d_k$$

then $\varepsilon_k^1 = \varepsilon_k^2$, for any $k = 1, 2, ..., m$, which verifies the theorem equality. ■

From the Proof above we can conclude that, in the input terminology, all the 1-dimensional chains that connect 2 given vertices: $I_p^0, I_q^0 \in$ □ ^{0} have the same length. This means that over the vertices set □ ^{0} of the abstract cubic complex $\mathcal{K}^n$, univocally, it is defined a function $d :$ □ $^0 \times$ □ $^0 \rightarrow$ □ $^+$ so, that for any two vertices $I_p^0, I_q^0 \in$ □ ^{0} it is held the following equality:

(1) $d\left(I_p^0,\ I_q^0\right) = \sum_{k=1}^{m} \varepsilon_k d_k$, where $d_k \in R^+$ represents the weight of the parallel edges class □ $_k$, $k = 1,2,...$, and

$$\varepsilon_k = \begin{cases} 0, & \text{if an arbitrary chain taken, which connects the } I_p^0, I_q^0, \\ & \text{intersects the } C_k \text{ class an even number of times} \\ 1, & \text{if an arbitrary chain taken, which connects the } I_p^0, I_q^0, \\ & \text{intersects the } C_k \text{ class an uneven number of times} \end{cases}$$

We denote by □ ^{1} the set of all the 1-dimensional cubes of the abstract cubic complex $\mathcal{K}^n$, and by □ $_1^1$ - a certain subset from □ ^{1}. Let F□ $_1^1$ be the set of all the cubes from $\mathcal{K}^n$ that contain as its facet at least one cube from □ $_1^1$. It is obvious that □ $_1^1 \subset F$□ $_1^1$.

We denote by $Vid(F$□ $_1^1)$ the union of vacuums of all the cubes from F□ $_1^1$.

Definition 4: *The $Vid(F$□ $_1^1)$ set, with the property that if we eliminate it from the abstract connected complex $\mathcal{K}^n$ we obtain two abstract cubic connected complexes, is called transversal of $\mathcal{K}^n$ and it is denoted by $T_{Q_1^1}(\mathcal{K}^n)$.*

If $\mathcal{K}^n$ is acyclic complex then it is obvious the following theorem:

Theorem 2: *Any class □ of parallel edges of the abstract cubic complex $\mathcal{K}^n$ determines one of the complex' transversal and is denoted by $T_□(\mathcal{K}^n)$.*

Theorem 3: *For the set of all 0-dimensional □ ^{0} cubes of the abstract cubic complex $\mathcal{K}^n$ the function defined by (1) represents an univocal metrics.*

Proof: First, it will be proved that the function defined by (1) verifies the following metrical properties:

1) as the weights of the parallel edges classes are real positive numbers, results that for any two elements: $I_i^0, I_j^0 \in$ □ ^{0} the following inequality takes place $d\left(I_i^0,\ I_j^0\right) \geq 0$. Let us show that $d\left(I_i^0,\ I_j^0\right) = 0$ if and only if $I_i^0 = I_j^0$.

a) If I_i^0 and I_j^0 coincide, i.e. these two vertices are not separated by any transversal, then $\varepsilon_k = 0,\ k = 1, 2, ..., m$. Thus, $d\left(I_i^0,\ I_j^0\right) = 0$.

b) If $d\left(I_i^0,\ I_j^0\right) = 0$, then, because the weights of the parallel edges classes are positive numbers, it results that $\varepsilon_k = 0,\ \forall k = 1, 2, ..., m$. Thus, the chain $L^1\left(I_i^0,\ I_j^0\right)$ intersects each class of parallel edges □ $_k$, $k = \overline{1,m}$, an even number of times. This means that the vertices I_i^0, I_j^0 coincide.

2) Let us demonstrate that for any two elements $I_i^0, I_j^0 \in \square^0$ takes place the symmetry metrical property: $d\left(I_i^0,\ I_j^0\right) = d\left(I_j^0,\ I_i^0\right)$. Suppose the contrary. This means that there exists an chain $L_1^1\left(I_i^0,\ I_j^0\right)$ with the value of the function (1) equal to $d_{L_1^1}\left(I_i^0,\ I_j^0\right)$ and an chain $L_2^1\left(I_j^0,\ I_i^0\right)$ with the value of the function (1) equal to $d_{L_2^1}\left(I_j^0,\ I_i^0\right)$, so that $d_{L_1^1}\left(I_i^0,\ I_j^0\right) \neq d_{L_2^1}\left(I_j^0,\ I_i^0\right)$.

We examine the chain $L' = L_1^1\left(I_i^0,\ I_j^0\right) \bigcup L_2^1\left(I_j^0,\ I_i^0\right)$, which is obviously a cycle. This cycle intersects each class of parallel edges $\square_k$, $k = \overline{1,m}$, an even number of times according to the hypotheses that the abstract cubic $\mathcal{K}^n$ complex is acyclic. Thus, each of the chains $L_1^1\left(I_i^0,\ I_j^0\right)$, $L_2^1\left(I_j^0,\ I_i^0\right)$ intersect each class $\square_k$, $k = \overline{1,m}$, either an even number of times, or an uneven number of times. Considering the function definition (1), as a result we obtain that $L' = L_1^1\left(I_i^0,\ I_j^0\right) = L_2^1\left(I_j^0,\ I_i^0\right)$.

3) Let I_i^0, I_j^0, I_s^0 be three different vertices in $\square^0$. It will be proved that the triangle inequality takes place:

$$d\left(I_i^0,\ I_j^0\right) \le d\left(I_i^0,\ I_s^0\right) + d\left(I_s^0,\ I_j^0\right).$$

We denote by $L^1\left(I_i^0,\ I_j^0\right), L^1\left(I_i^0,\ I_s^0\right), L^1\left(I_s^0,\ I_j^0\right)$ the chains that connect the couples of vertices and that have equal lengths with $d\left(I_i^0,\ I_j^0\right), d\left(I_i^0,\ I_s^0\right), d\left(I_s^0,\ I_j^0\right)$. We form the chain $L_1^1\left(I_i^0,\ I_j^0\right) = L^1\left(I_i^0,\ I_s^0\right) \bigcup L^1\left(I_s^0,\ I_j^0\right)$. Let d_{ij}^1 be the (1)-function's value determined by the chain $L_1^1\left(I_i^0,\ I_j^0\right)$. If we use similar notations for the other cases, i.e. $d_{is} = d\left(I_i^0,\ I_s^0\right)$ and $d_{sj} = d\left(I_s^0,\ I_j^0\right)$, then we have:

$$d_{ij}^1 \le d_{is} + d_{sj}.$$

The union $L^1\left(I_i^0,\ I_j^0\right) \bigcup L_1^1\left(I_i^0,\ I_j^0\right)$ is a cycle, which intersects each class of parallel edges an even number of times, because, according to the hypotheses that the $\mathcal{K}^n$ complex is acyclic. This means that each of the chains $L^1\left(I_i^0,\ I_j^0\right)$ and $L_1^1\left(I_i^0,\ I_j^0\right)$ intersect each class of parallel edges $\square_k$, $k = \overline{1,m}$, an even number of times or both of them an uneven number of times, that lead us to the following relation:

$$d_{ij} = d_{ij}^1 \le d_{is} + d_{sj}$$

Thus, the property 3) takes place. The metrical uniqueness results from the theorem proved above. ■

Repherences:

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