CUBIC DIFFERENTIAL SYSTEMS WITH SIX REAL INVARIANT STRAIGHT LINES ALONG TWO DIRECTIONS

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Sunt clasificate sistemele cubice cu exact şase drepte invariante de două direcţii ţinându-se cont la enumerare de gradul lor de invarianţă. Se arată că, din punct de vedere topologic, sunt 11 clase distincte de astfel de sisteme. Pentru fiecare dintre aceste clase este construit pe discul Poincaré portretul fazic.

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1 Introduction

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x, y),\ \frac{dy}{dt} = Q(x, y), \tag{1}$$

where P, $Q \in \mathbb{R}[x, y]$, and the polynomial vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y} \tag{2}$$

corresponding to system (1).

Denote $n = max\{deg(P), deg(Q)\}$. If $n = 2\,(n = 3)$ then system (1) is called quadratic (cubic).

The function $f : D \subset \mathbb{R}^2 \to \mathbb{C}$, $f \neq const$, is said to be an *elementary invariant* (or a *Darboux invariant*) for (2) if there exists a polynomial $K_f \in \mathbb{C}[x, y]$, $deg(K_f) \leq n - 1$ such that the identity

$$X(f) \equiv f(x, y)K_f(x, y)$$

holds. The polynomial K_f is called the *cofactor* of f. Denote by I_X the set of all elementary invariants of (2); $I_a = \{f \in \mathbb{C}[x, y] \,|\, f \in I_X\}$, $I_e = \{exp(\frac{g}{h}) \,|\, g, h \in \mathbb{C}[x, y],\ GCD(g, h) = 1,\ exp(\frac{g}{h}) \in I_X\}$. The elements from I_a (I_e) are called *algebraic invariants* (*exponential invariants*) of (2). In [1] it is shown that if $f = exp(g/h) \in I_e$, $h \neq const$, then $h \in I_a$ and $X(f) = gK_h + hK_f$.

Let $f \in \mathbb{C}[x, y]$ and $f = f_1^{n_1} \cdots f_s^{n_s}$ be its factorization in irreducible factors over $\mathbb{C}[x, y]$. Then $f \in I_a$ if and only if $f_j \in I_a, j = \overline{1, s}$. Moreover, $K_f = n_1K_{f_1} + \cdots + n_sK_{f_s}$. If $f_j \in I_a \bigcup I_e$, $\lambda_j \in \mathbb{C}$, $j = \overline{1, s}$, then $f_1^{\lambda_1} \cdots f_s^{\lambda_s} \in I_X$.

We will say that an algebraic invariant $f \in I_a$ has the *degree of invariance* equal to m, if m is the greatest positive integer such that f^m divides $X(f)$. For invariant straight lines $ax + by + c = 0$, $ax + by + c \in I_a$, such a definition was brought in [2]. If $f \in I_a$ has the degree of invariance equal to $m \geq 2$, then $exp(1/f), ..., exp(1/f^{m-1}) \in I_e$.

We say that the system (1) is *Darboux integrable* if there exists a non-constant function of the form

$$f = f_1^{\lambda_1} \cdots f_s^{\lambda_s}, \tag{3}$$

where $f_j \in I_a \bigcup I_e$ and $\lambda_j \in \mathbb{C}$, $j = \overline{1, s}$, such that either $f = const$ is a first integral (i.e. $K_f \equiv 0$) or f is an integrating factor (i.e. $K_f \equiv -\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}$) for (1). It can be shown that (3) is a first integral (an integrating factor) for (1) if and only if

$$\lambda_1K_{f_1}(x, y) + \cdots + \lambda_sK_{f_s}(x, y) \equiv 0$$
$$(\lambda_1K_{f_1}(x, y) + \cdots + \lambda_sK_{f_s}(x, y) \equiv -\tfrac{\partial P}{\partial x} - \tfrac{\partial Q}{\partial y}).$$

The purpose of this paper is to present the begining of qualitative investigation of the cubic system with six invariant straight lines.

Although the straight lines are the most simple representatives in the class of algebraic curves, the study of the differential equations with invariant straight lines is far from completion. It has attracted the attention of many researches and at present there are a lot of papers devoted to this subject. So, in [3-7] for different classes of polynomial systems conditions for the existence of invariant straight lines are obtained.

A set of invariant straight lines can be infinite, finite or empty. Systems with infinite number of invariant straight lines will not be considered.

In papers [8-14] the estimation for the number of invariant straight lines is given. Denote by $\alpha(n)$ the maximum number of the invariant straight lines and by $\beta(n)$ the maximum number of slopes of this lines in the class of n-polynomial differential systems. In [8] it is shown that $\alpha(2) = 5$; in [9,10] - $\alpha(3) = 8$; in [10,11,12] - $\alpha(4) = 9$; in [13] - $\alpha(5) = 14$ and that $2n+1+\frac{1-(-1)^n}{2} \le \alpha(n) \le 3n-1, n > 5$; in [14] - $\beta(3) = 6, \beta(4) = 9$ and in [15] - $\beta(n) = \alpha(n-1) + 1$.

The problem of coexistence of invariant straight lines and limit cycles were investigated in [16-25]. As follows from [16-21], a quadratic system with at least two invariant straight lines has no limit cycles and with one invariant straight line can have at most one limit cycle. A cubic system with at least five real invariant straight lines has no limit cycles [21,22]. The same system with four real or with two real and two complex conjugate invariant straight lines can have at most one limit cycle [23-25]. A cubic system with four complex conjugate invariant straight lines can have two limit cycles [25], examples with more than two limit cycles are not known.

The problem of the center for cubic differential systems with four and three invariant straight lines is investigated in [2,26-29]. According to [2] ([26-29]) the cubic differential system with a weak focus at $(0,0)$ and at least four (three) invariant straight lines has a center at the origin of coordinates if and only if the first two (seven) focal values vanish.

A qualitative investigation of cubic systems with exactly eight and exactly seven invariant straight lines was carried out in [9,30,31]. In this paper a similar qualitative investigation is done for cubic differential systems with exactly six real invariant straight lines along two directions.

The main obtained results are shown in the following theorem:

Theorem. *Any cubic system having real invariant straight lines along two directions with total degree of invariance six via affine transformation and time rescaling can be written as one of the following eight systems. In the figure associated to each system is presented the phase portrait in the Poincaré disc.*

$$\begin{cases} \dot{x} = x(x+1)(x-a),\ a>0, & \\ \dot{y} = \beta y(y+1)(y-b),\ b>0, & Fig.1(\beta<0), \\ \beta(|\beta-1|+|b-a|)(|\beta-a^2|+|b-\frac{1}{a}|)\neq 0, & Fig.2(\beta>0); \end{cases} \quad (4)$$

$$\begin{cases} \dot{x} = x^2(x+1), & Fig.3(\beta<0), \\ \dot{y} = \beta y(y+1)(y-b),\ b|\beta|>0, & Fig.4(\beta>0); \end{cases} \quad (5)$$

$$\begin{cases} \dot{x} = x^3, & \\ \dot{y} = -y(y+1)(y-b),\ b>0, & Fig.5; \end{cases} \quad (6)$$

$$\begin{cases} \dot{x} = x^3, & \\ \dot{y} = y(y+1)(y-b),\ b>0, & Fig.6; \end{cases} \quad (7)$$

$$\begin{cases} \dot{x} = x^2(x+1), & Fig.7(\beta<0), \\ \dot{y} = \beta y^2(y+1),\ \beta(\beta-1)\neq 0, & Fig.8(\beta>0); \end{cases} \quad (8)$$

$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = -y^2(y+1), \end{cases} \qquad Fig.9; \tag{9}$$

$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = y^2(y+1), \end{cases} \qquad Fig.10; \tag{10}$$

$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = -y^3. \end{cases} \qquad Fig.11. \tag{11}$$

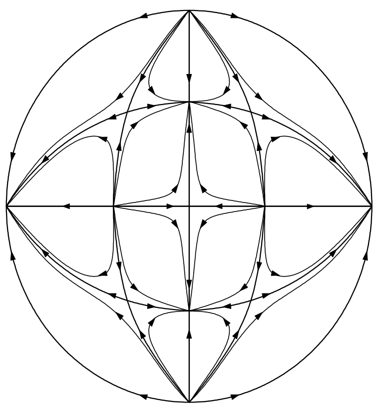


Fig.1

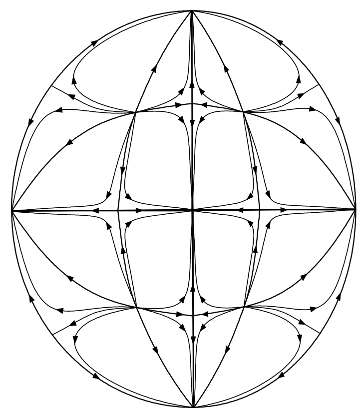
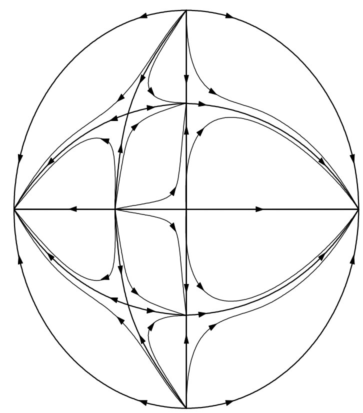


Fig.2



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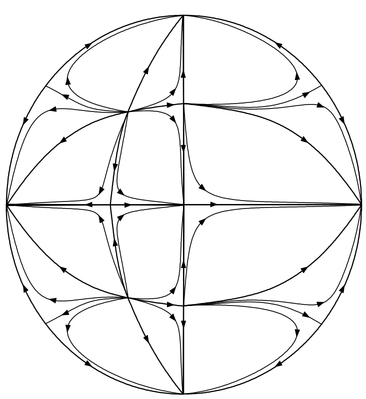


Fig.4

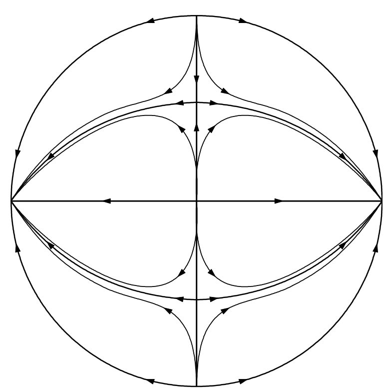


Fig.5

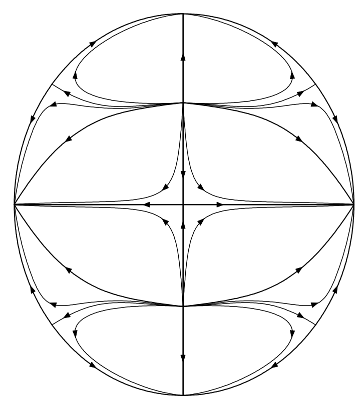


Fig.6

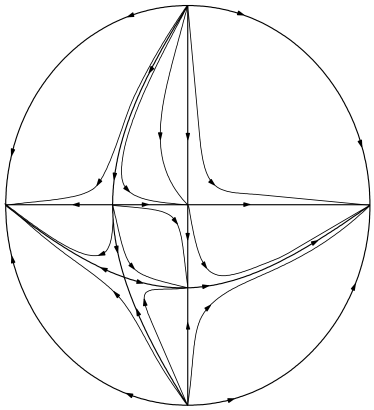


Fig.7

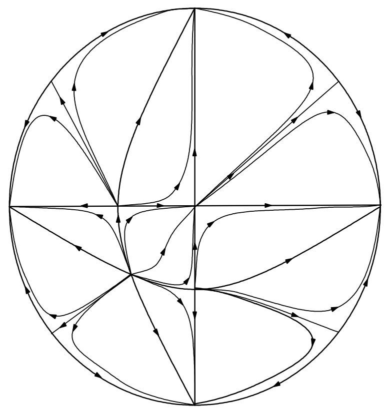


Fig.8

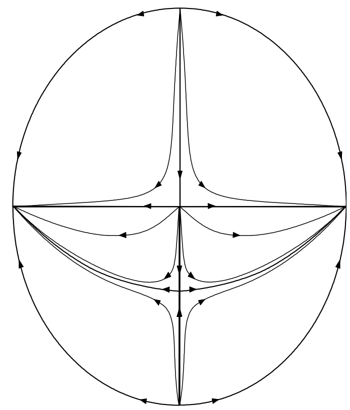


Fig.9

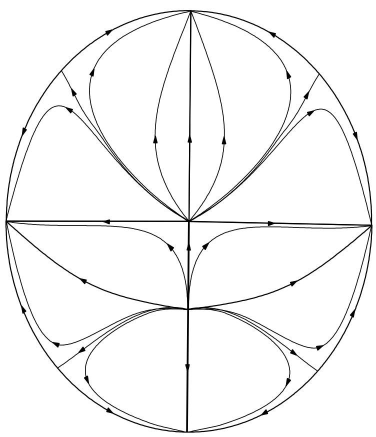


Fig.10

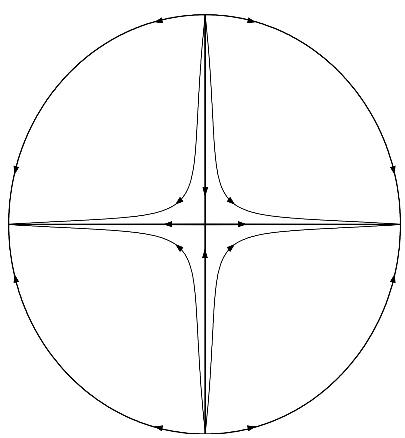


Fig.11

2 Preliminaries

We consider the real cubic differential system

$$\frac{dx}{dt} = \sum_{r=0}^{3} P_r(x,y), \qquad \frac{dy}{dt} = \sum_{r=0}^{3} Q_r(x,y), \tag{12}$$

where $P_r(x,y) = \sum\limits_{j+l=r} a_{jl}x^j y^l,\ Q_r(x,y) = \sum\limits_{j+l=r} b_{jl}x^j y^l$. Assume that the members from the right-hand side of system (12) have not a non-constant common factor.

We mention here some properties of system (12):

a) in the finite part of the phase plane system (12) has at most nine singular points;

b) at infinity the system (12) has at most four singular points if $yP_3(x,y) - xQ_3(x,y) \not\equiv 0$. In case $yP_3(x,y) - xQ_3(x,y) \equiv 0$ the infinity is degenerate, i.e. consists only from singular points;

c) the system (12) has in the finite part of the plane not more than eight invariant straight lines;

d) the infinite line represents an invariant straight line for (12);

e) the system (12) has invariant straight lines along at most six different directions;

f) the system (12) cannot have more than three parallel invariant straight lines.

Let $a_j x + b_j y + c_j = 0,\ j = 1,2,\ a_1 b_2 - a_2 b_1 \neq 0$ be two real invariant straight lines of system (12). The transformation $x_1 = a_1 x + b_1 y + c_1,\ y_1 = a_2 x + b_2 y + c_2$ reduces (12) to a system of the

Lotka-Volterra form (we keep the old notations for variables)

$$\begin{cases} \dot{x} = x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} = y(b_{01} + b_{11}x + b_{02}y + b_{21}x^2 + b_{12}xy + b_{03}y^2). \end{cases} \quad (13)$$

The property f) says that every cubic system with at least four real invariant straight lines can be written in the form (13).

A straight line $y = Ax + B,\ A \neq 0$ is invariant for system (13) if and only if A and B are the solutions of the system:

$$\begin{array}{l} B(b_{01} + b_{02}B + b_{03}B^2) = 0, \\ b_{11}B + b_{12}B^2 + [b_{01} - a_{10} + (2b_{02} - a_{11})B + (3b_{03} - a_{12})B^2] \cdot A = 0, \\ b_{21}B + [b_{11} - a_{20} + (2b_{12} - a_{21})B] \cdot A + [b_{02} - a_{11} + (3b_{03} - 2a_{12})B] \cdot A^2 = 0, \\ b_{21} - a_{30} + (b_{12} - a_{21}) \cdot A + (b_{03} - a_{12}) \cdot A^2 = 0. \end{array} \quad (14)$$

The cofactor of this line is

$$K(x, y) = c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2,$$

where

$$c_{00} = b_{01} + b_{02}B + b_{03}B^2,\ c_{10} = b_{11} + b_{12}B + (b_{02} - a_{11})A + (2b_{03} - a_{12})AB,\ c_{01} = b_{02} + b_{03}B,$$

$$c_{20} = b_{21} + (b_{12} - a_{21})A + (b_{03} - a_{12})A^2,\ c_{11} = b_{12} + (b_{03} - a_{12})A,\ c_{02} = b_{03}.$$

3 Canonical forms and Darboux integrability

There are the following configurations of six invariant straight along two directions:

$$1)\ (3,3),\ Fig.\,12a);\quad 2)\ (3(2),3),\ Fig.\,12b);\quad 3)\ (3(3),3),\ Fig.\,12c);$$

$$4)\ (3(2),3(2)),\ Fig.\,12d);\quad 5)\ (3(3),3(2)),\ Fig.\,12e);\quad 6)\ (3(3),3(3)),\ Fig.\,12f).$$

Notation $(3(2), 3)$ means that along of one direction there are two distinct straight lines from which one is double (i.e. has degree of invariance equal to two), and along of the second direction there are three distinct invariant straight lines; $(3(3), 3(2))$ means that along of one direction the differential system has one triple invariant straight line, and along of the second direction there are two distinct invariant straight lines from which one is double and so on.

If an invariant straight line has multiplicity $m > 1$, then the number m appears near the corresponding straight line and this line is more thick.

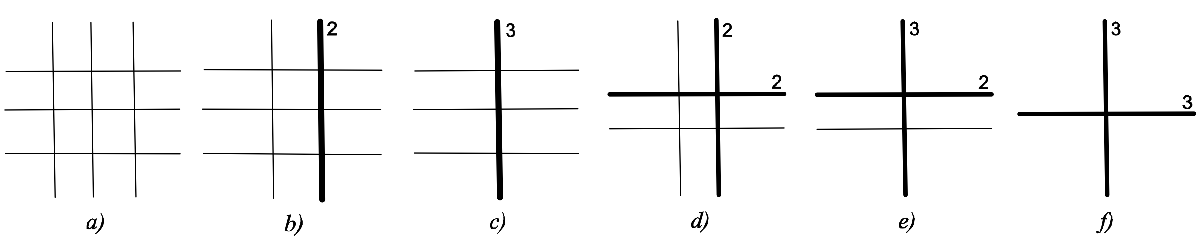


Fig.12

The cubic systems that realize configurations $1) - 6)$ via affine transformations and time rescaling can be written in the following form, respectively:

$$\begin{cases} \dot{x} = x(x+1)(x-a), \quad a > 0, \\ \dot{y} = \beta y(y+1)(y-b), \; b > 0, \beta \neq 0; \end{cases} \tag{15}$$

$$\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = \beta y(y+1)(y-b), \; b > 0, \beta \neq 0; \end{cases} \tag{16}$$

$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = \beta y(y+1)(y-b), \; b > 0, \beta \neq 0; \end{cases} \tag{17}$$

$$\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = \beta y^2(y+1), \; \beta \neq 0; \end{cases} \tag{18}$$

$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = \beta y^2(y+1), \; \beta \neq 0; \end{cases} \tag{19}$$

$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = \beta y^3, \; \beta \neq 0. \end{cases} \tag{20}$$

The systems (15)-(20) are Darboux integrable and have respectively first integrals:

$$[x^{1/a}(x+1)^{-1/(a+1)}(x-a)^{-1/(a(a+1))}]^{\beta b(b+1)} y^{-b-1}(y+1)^b(y-b) = C;$$

$$[x(x+1)^{-1} exp(1/x)]^{\beta b(b+1)} y^{-b-1}(y+1)^b(y-b) = C;$$

$$y^{-2(b+1)}(y+1)^{2b}(y-b)^2 exp(\beta b(b+1)/x^2) = C;$$

$$x^{-1}(x+1)^{\beta} y(y+1)^{-1} exp(-\beta/x) exp(1/y) = C;$$

$$y^{-2}(y+1)^2 exp(\beta/x^2) exp(-2/y) = C;$$

$$x^{-2}y^{-2}(\beta y^2 - x^2) = C.$$

To emphasize the cases when (15)-(20) contain more than six invariant straight lines we use the algebraic systems of equation (14). Thus, writing system (14) in condition (15) and solving it for A and B, we obtain that (15) has exactly seven invariant straight lines if and only if one of the following two series of conditions hold $\beta - 1 = b - a = 0$, $a \neq 1$ and $\beta - a^2 = b - \frac{1}{a}$, $a \neq 1$, that is, when (15) has one of the forms:

$$\begin{cases} \dot{x} = x(x+1)(x-a), \\ \dot{y} = y(y+1)(y-a), a > 0, a \neq 1; \end{cases} \tag{21}$$

$$\begin{cases} \dot{x} = x(x+1)(x-a), \\ \dot{y} = a^2 y(y+1)(y-\frac{1}{a}), a > 0, a \neq 1. \end{cases} \tag{22}$$

For (21) ((22)) the invariant straight lines $l_j = 0$, $j = \overline{1,7}$ are

$$l_1 = x,\ l_2 = x+1,\ l_3 = x-a,\ l_4 = y,\ l_5 = y+1,\ l_6 = y-a\,(l_6 = y - \tfrac{1}{a}),\ l_7 = y - x\,(l_7 = y + \tfrac{1}{a}x).$$

We mention that system (22) can be reduced to system (21) by substitution $x \to x$, $y \to -y/a$.

The system (15) has eight invariant straight lines if and only if $\beta = a = b = 1$, that is when it has the form

$$\begin{cases} \dot{x} = x(x+1)(x-1), \\ \dot{y} = y(y+1)(y-1). \end{cases}$$

The invariant straight lines are:

$$l_1 = x,\ l_{2,3} = x \pm 1,\ l_4 = y,\ l_{5,6} = y \pm 1,\ l_{7,8} = y \pm x.$$

Now it becomes clear why in (4) the condition $(|\beta - 1| + |b - a|)(|\beta - a^2| + |b - \frac{1}{a}|) \neq 0$ is imposed.

The equalities (14) show us that systems (16)$-$(19) cannot have eight invariant straight lines and systems (16), (17), (19) and (20) cannot have exactly seven invariant straight lines.

The system (18) has exactly seven invariant straight lines if and only if $\beta = 1$, i.e.

$$\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y^2(y+1). \end{cases}$$

The invariant straight lines are

$$l_1 = l_2 = x,\ l_3 = x + 1,\ l_4 = l_5 = y,\ l_6 = y + 1,\ l_7 = y - x.$$

The system (20) has eight invariant straight lines if and only if $\beta > 0$. In this case substitutions $x \to \sqrt{\beta}x$, $y \to y, t \to t/\beta$ reduce (20) to a system

$$\dot{x} = x^3, \quad \dot{y} = y^3,$$

with $l_{1,2,3} = x$, $l_{4,5,6} = y$, $l_{7,8} = y \pm x$. If $\beta < 0$, then substitutions $x \to \sqrt{-\beta}x$, $y \to y$, $t \to -t/\beta$ reduce (20) to a system (11).

By the some substitutions the system (17) can be reduced to one of systems (6), (7).

4 The phase portraits

We denote by SP $-$ singular points; λ_1 and λ_2 the eigenvalues of SP; S $-$ saddle ($\lambda_1\lambda_2 < 0$), TS $-$ topological saddle; N^s $-$ stable node (λ_1, $\lambda_2 < 0$), N^u $-$ unstable node (λ_1, $\lambda_2 > 0$), $DN^{s(u)}$ $-$ stable (unstable) dicritical node ($\lambda_1 = \lambda_2 \neq 0$), $TN^{s(u)}$ $-$ stable (unstable) topological node; S-$N^{s(u)}$ $-$ saddle-node with stable (unstable) parabolic sector; $P^{s(u)}$ $-$ stable (unstable) parabolic sector; H $-$ hyperbolic sector.

4.1 Infinity

In case of systems (7) and (10) ((6), (9) and (11)) it is convenient to consider $\beta = 1$ (respectively $\beta = -1$). Then systems (4)$-$(11) for which $\beta < 0$ have at the infinity only two real singular points, and for $\beta > 0$ have four such points. Singular points, with eigenvalues and their type are given in Tab.1.

Tab.1

SP	$\lambda_1;\ \lambda_2$	$\beta < 0$	$\beta > 0$
$(1,0,0)$	$-1;\ -1$	DN^s	DN^s
$(0,1,0)$	$-\beta;\ -\beta$	DN^u	DN^s
$(1,-\frac{1}{\sqrt{\beta}},0)$	$-1;\ 2$	$-$	S
$(1,\frac{1}{\sqrt{\beta}},0)$	$-1;\ 2$	$-$	S
		$Fig.13a)$	$Fig.13b)$

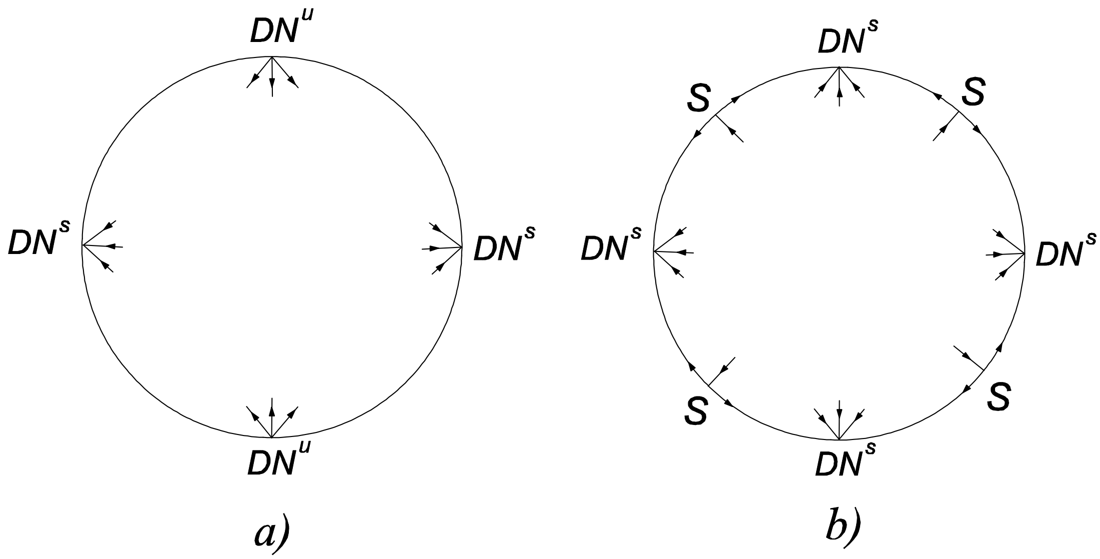


Fig.13 *a)* for $\beta < 0$, *b)* for $\beta > 0$.

4.2 System (4)

For system (4) the results of qualitative investigation of singular points in the finite part of the phase plane are given in Tab.2.

Tab.2

SP	λ_1; λ_2	$\beta < 0$	$\beta > 0$
$(0, 0)$	$-a$; $-\beta b$	S	N^s
$(0, b)$	$-a$; $\beta b(b + 1)$	N^s	S
$(-1, b)$	$a + 1$; $\beta b(b + 1)$	S	N^u
$(-1, 0)$	$a + 1$; $-\beta b$	N^u	S
$(-1, -1)$	$a + 1$; $\beta(b + 1)$	S	N^u
$(0, -1)$	$-a$; $\beta(b + 1)$	N^s	S
$(a, -1)$	$a(a + 1)$; $\beta(b + 1)$	S	N^u
$(a, 0)$	$a(a + 1)$; $-\beta b$	N^u	S
(a, b)	$a(a + 1)$; $\beta b(b + 1)$	S	N^u
		Fig.1	*Fig*.2

4.3 System (5)

For system (5) the results of qualitative investigation of singular points in the finite part of the phase plane are given in Tab.3.

Tab.3

SP	$\lambda_1;\ \lambda_2$	$\beta < 0$	$\beta > 0$
$(0,0)$	$0;\ -\beta b$	S-N^u	S-N^s
$(0,b)$	$0;\ \beta b(b+1)$	S-N^s	S-N^u
$(-1,b)$	$1;\ \beta b(b+1)$	S	N^u
$(-1,0)$	$1;\ -\beta b$	N^u	S
$(-1,-1)$	$1;\ \beta(b+1)$	S	N^u
$(0,-1)$	$0;\ \beta(b+1)$	S-N^s	S-N^u
		$Fig.3$	$Fig.4$

To establish the type of singular points with $\lambda_1 = 0$ we used the theorem 2, p.87 from [32].

4.4 Systems (6) and (7)

For system (6) ((7)) we have Tab.4.

Tab.4

SP	$\lambda_1;\ \lambda_2$	(6), $\beta = -1$	(7), $\beta = 1$
$(0,0)$	$0;\ -\beta b$	TN^u	TS
$(0,b)$	$0;\ \beta b(b+1)$	TS	TN^u
$(0,-1)$	$0;\ \beta(b+1)$	TS	TN^u
		$Fig.5$	$Fig.6$

4.5 System (8):

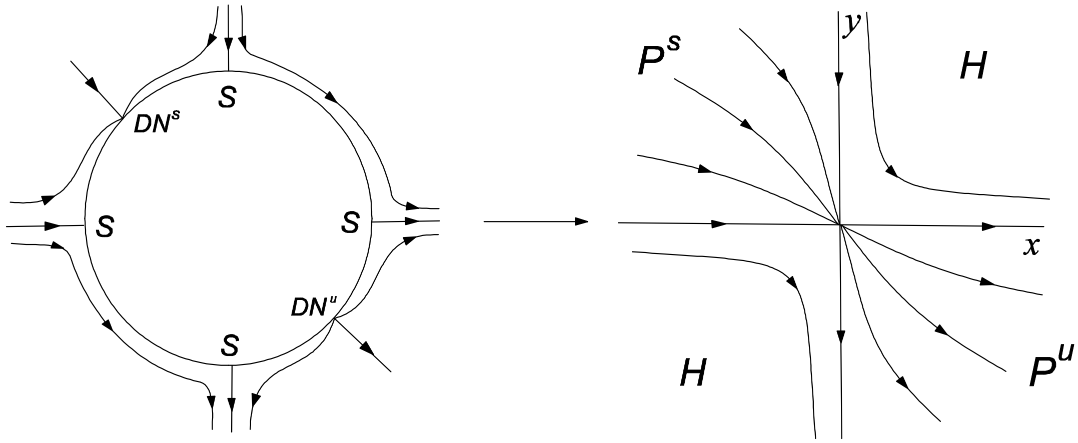
Tab.5

SP	$\lambda_1;\ \lambda_2$	$\beta < 0$	$\beta > 0$	SP	$\lambda_1;\ \lambda_2$	$\beta < 0$	$\beta > 0$
$(0,0)$	$0;\ 0$	HP^sHP^u	P^uHP^sH	$(-1,-1)$	$1;\ \beta$	S	N^u
$(-1,0)$	$1;\ 0$	S-N^u	S-N^u	$(0,-1)$	$0;\ \beta$	S-N^s	S-N^u
$\beta < 0 : Fig.7;$				$\beta > 0 : Fig.8$			

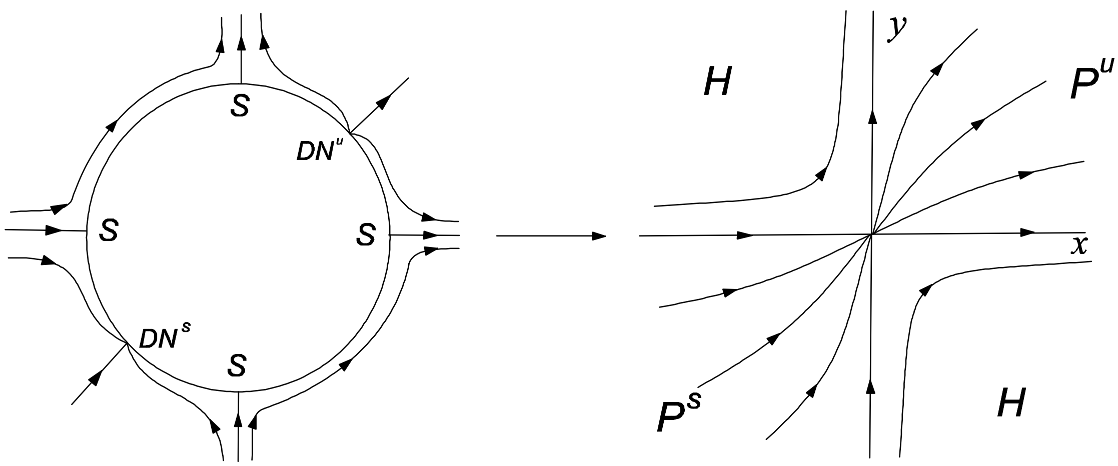
For system (8) a singular point $(0,0)$ has both eigenvalues null. To determine the behavior of trajectories in the neighborhood of $(0,0)$, we write (8) in the polar coordinates $x = \rho cos\theta, y = \rho sin\theta$:

$$\begin{cases} \frac{d\rho}{d\tau} = \rho(\rho cos^4\theta + \beta\rho sin^4\theta + cos^3\theta + \beta sin^3\theta), \\ \frac{d\theta}{d\tau} = sin\theta cos\theta(\beta\rho sin^2\theta - \rho cos^2\theta + \beta sin\theta - cos\theta), \end{cases} \tag{23}$$

where $\tau = \rho t$. System (23) has the following singular points with the first coordinate ρ equal to zero and the second one belonging to $[0,\ 2\pi]$: $M_1(0,0)$, $M_2(0,\pi)$, $M_3(0,\pi/2)$, $M_4(0,3\pi/2)$, $M_5(0, arctan\frac{1}{\beta})$ and $M_6(0,\pi + arctan\frac{1}{\beta})$. For M_1 and M_2 we have $\lambda_{1,2} = \pm 1$; for M_3 and M_4: $\lambda_{1,2} = \pm\beta$; for M_5: $\lambda_1 = \lambda_2 = \beta/\sqrt{1+\beta^2}$ and for M_6: $\lambda_1 = \lambda_2 = -\beta/\sqrt{1+\beta^2}$ $(Fig.14, 15)$.



$Fig.14$ $(\beta < 0)$



$Fig.15$ $(\beta > 0)$

4.6 Systems (9) and (10) $(Fig.\,9,\ 10)$

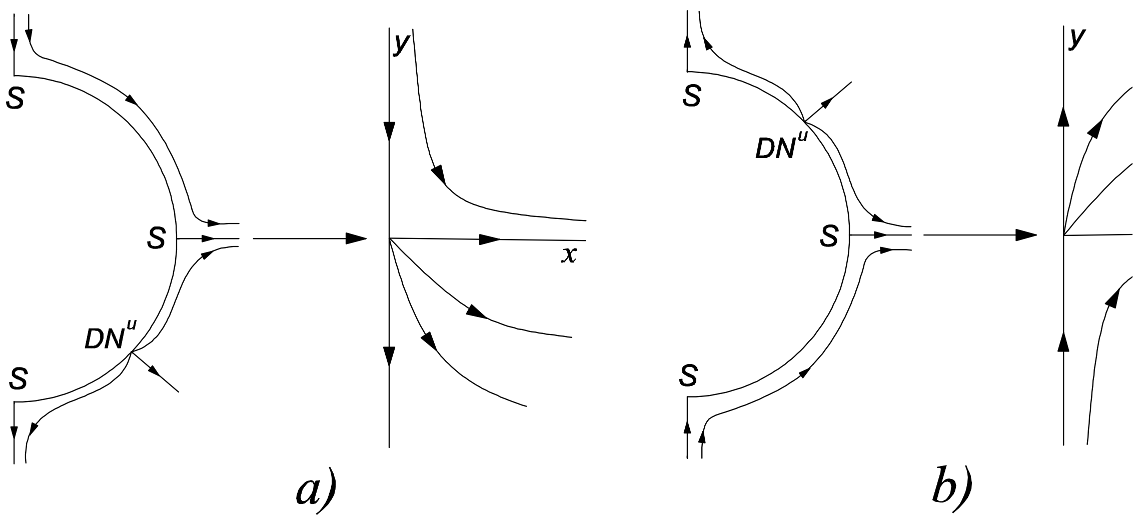
We consider the system (19) in which $\beta = -1$ or $\beta = 1$. This system is symmetric with respect to the y-axis. It has singular points $(0,0)$ and $(0,-1)$. Using theorem 2, p.87 from [32] it is easily determined that $(0,-1)$ is a saddle if $\beta < 0$ and it is an unstable nod if $\beta > 0$. To establish the behavior of trajectories in the neighborhood of singular point $(0,0)$ of (19) we consider $x \geq 0$ and make the substitution $X = x^2, y = y$:

$$\dot{X} = 2X^2, \quad \dot{y} = \beta y^2(y+1), \ X \geq 0. \tag{24}$$

In polar coordinates $X = \rho cos\theta$, $y = \rho sin\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\tau = \beta t$ the system (24) we can be writen in the form

$$\begin{cases} \frac{d\rho}{d\tau} = \rho(\beta\rho sin^4\theta + \beta sin^3\theta + 2cos^3\theta), \\ \frac{d\theta}{d\tau} = sin\theta cos\theta(\beta\rho sin^2\theta + \beta sin\theta - 2cos\theta). \end{cases}$$

The singular point $(0,0)$ has the eigenvalues $\lambda_{1,2} = \pm 2$; $(0,-\pi/2)$ and $(0,\pi/2)$: $\lambda_{1,2} = \pm\beta$; $(0, \pm arctan\frac{2}{\beta})$: $\lambda_{1,2} = \frac{2|\beta|}{\sqrt{4+\beta^2}}$ $(Fig.16)$.



$Fig.16$ *a)* for $\beta < 0$, *b)* for $\beta > 0$

4.7 System (11)

The given system is symmetric with respect to the origin of coordinates. In polar coordinates it can be written as $\dot{\rho} = \rho^3 cos2\theta,\ \dot{\theta} = -\frac{1}{2}\rho^2 sin2\theta$ and has the first integral $\rho sin2\theta = C$. $(Fig.(11))$.

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