STABILITY OF THE FULL SET OF STACKELBERG EQUILIBRIUM IN THE DYNAMIC GAME OF THREE PLAYERS WITH THREE STAGES

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Pentru modelarea matematică a proceselor decizionale în sisteme cu structuri ierarhice se pot utiliza jocurile dinamice pe multe etape. La fiecare etapa a jocului, jucătorii aleg strategiile sale din mulţimea de răspunsuri optimale, determinate de alegerea strategiilor de către partenerii la joc. Folosind metoda inducţiei recursive situaţiile Stackelberg de echilibru se determină prin soluţionarea unei probleme de maxmin pe trei nivele. Se defineşte mulţimea de situaţii complet Stackelberg de echilibru în cazul existenţei erorilor minimal şi maximal admisibile la efectuarea operaţiilor matematice. Pentru aceste probleme sunt prezentate condiţiile în baza cărora se construiesc mulţimile de stabilitate cînd mulţimile de startegii admisibile şi funcţiile scop ale jucătorilor sunt perturbate.

Let us consider the dynamic game of three players with three stages in the following strategic form

$$\Gamma = \langle Z, Y, X; G : Z \times Y \to R, F : Z \times Y \times X \to R, H : Z \times Y \times X \to R \rangle .$$

where Z, Y, X are the sets of available strategies and G, F, H are the payoff functions for the player 1, player 2 and player 3 respectively .

The game occurs as follows: in the first stage the player 1 chooses independently his strategy $z \in Z$ and communicates this strategy to the player 2. In the second stage the player 2 observes a chosen strategy of player 1 and chooses independently his strategy $y \in Y$, after than communicates this pair of strategy (z, y) to the player 3. In the third stage the player 3 observes the pair of strategy (z, y) and chooses his strategy $x \in X$. After this the game is considered finished. Also suppose that the player 3 wants to maximize their payoff function and the players 1 an 2 want to used the maximin optimally principle (guaranteed results).

This class of dynamic game is used for mathematical modelling of decision making problems in economical systems with the hierarchical structure. The class of two players dynamic game with 2 stages where the players use the maximin optimally principle is also called " the Germeier games" [1,2]

Similarly to strategic form game, the equilibrium concept in dynamic games is based on the idea that at every stage of the game each player plays the best response to the play of other players.

Suppose that the all mathematical operations are executed without errors. Then the solution of the dynamic game of three players with three stages can be found by the following backward induction algorithm:

A) the player 3 observes the chosen strategies $z \in Z$ by player 1 and the strategy $y(z)$ by player 2 and he will choose his strategy from the optimal reaction set of player 3 to a strategy $z \in Z$ and $y(z) \in Y$, so he will choose the following best response function

$$x^*(z, y(z)) \in R(z, y) = Arg \max_{x \in X} H(z, y(z), x) =$$
$$= \left\{ x^*(z, y(z)) \in X : \max_{x \in X} H(z, y(z), x) = H(z, y(z), x^*(z, y(z))) \right\};$$

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B) the player 2 knowing that the player 3 will play the strategy $x^*(z, y(z))$ chooses the strategy from the optimal reaction set of player 2 for a strategy $z \in Z$ of player 1, so he will choose the following best response function

$$y^*(z) \in L(z) = Arg \max_{y \in Y} \min_{x^*(z,y(z)) \in R(z,y)} F(z, y, x^*(z, y)) =$$

$$= \left\{ y^*(z) \in Y : \max_{y \in Y} \min_{x^*(z,y(z)) \in R(z,y)} F(z, y, x^*(z, y(z))) = F(z, y^*(z), x^*(z, y^*(z))) \right\};$$

C) the player 1 knowing that the player 2 will choose the strategy $y^*(z)$ and player 3 will choose the strategy $x^*(z, y^*(z))$ chooses the strategy from the following set of best response reaction

$$z^* \in Z^* \equiv Arg \max_{z \in Z} \min_{y^*(z,)) \in L(z)} G(z, y^*(z)) =$$

$$= \left\{ z^* \in Z : \max_{z \in Z} \min_{y^*(z)) \in L(z)} G(z, y^*(z)) = G(z^*, y^*(z^*)) \right\};$$

Let us introduce the following definition.

Definition 1 *The strategy profiles $(z^*, y^*, x^*) \equiv (z^*, y^*(z^*), x^*(z^*, y^*(z^*))$ defined by the steps A)-C) of the backward induction algorithm are called the Stackelberg equilibrium profile in the dynamic game of three players with three stages.*

Let $SE(\Gamma)$ denote the set of the Stackelberg equilibrium profiles in the dynamic game of three players with three stages.

Using the results from [3] we can prove the following theorem.

Theorem 2 *Let the dynamic game of three players with three stages in the strategic form Γ satisfies the following conditions:*
1. *the sets of strategies Z, Y, X are compact sets from R^n, R^m and R^k respectively;*
2. *the functions $H(z, y, x), F(z, y, x)$ and $G(z, y)$ are continuous on the sets Z, Y and X;*
3. *point to set mapping $R(z, y)$ are Hausdorff continuous on Y for all $z \in Z$, and the point to set mapping $L(z)$ are Hausdorff continuous on Z.*
Then exists the Stackelberg equilibrium profiles in the dynamic game of three players with three stages, i.e. $SE(\Gamma) \neq \emptyset$.

In the definition of problem stability [4] is supposed that the description of mathematical model is approximative, but the required mathematical operations are performed exactly. Then the investigation of stability problems is to establish if for sufficient small perturbations of mathematical model the perturbation of solutions will be also sufficient small.

Here we use the notion of stability described in [5]. The main idea of this notion consists in the following: suppose that the required mathematical operations are performed with errors and it is necessary to establish the relations between errors of operations and perturbations of the model such that the required conditions of stability are satisfied. In this paper we use the described idea to study the stability of the Stackelberg equilibrium profiles in the dynamic game of three players with three stages.

For solving any problems using the computers there are "minimal available errors" denoted by top index l and "maximal available errors" denoted by top index r. According the steps A)-C) we suppose that for determining the Stackelberg equilibrium profiles in the dynamic game of three players with three stages the following errors are used:

- let $\alpha_x \geq 0$, $\alpha_x^r \geqslant 0$ and $\alpha_x^l \geqslant 0$ denote simple, maximal and minimal available errors respectively in calculations of the "$\max\limits_{x\in X}$" operation to determine the best response function $x^*(z, y(z))$ of player 3 as in A);
- let $\alpha_y \geq 0$, $\alpha_y^r \geqslant 0$ and $\alpha_y^l \geqslant 0$ denote respectively simple, maximal and minimal available errors in calculations of the "$\max\limits_{y\in Y} \min\limits_{x^*(z,y(z))\in R(z,y)}$" operation to determine the best response function $y^*(z)$ of player 2 as in B);
- let $\alpha_z \geq 0$, $\alpha_z^r \geqslant 0$ and $\alpha_z^l \geqslant 0$ denotes respectively simple, maximal and minimal available errors in calculations of the "$\max\limits_{z\in Z} \min\limits_{y^*(z,))\in L(z)}$" operation to determinate the best response reaction $z^* \in Z^*$ of player 1 as in C).

The interpretation of these errors: the maximal errors give the worse precision of the solution, the minimal errors give the best precision of the solution.

Also, denote by $\varepsilon = (\varepsilon_z, \varepsilon_y, \varepsilon_x)$ the errors which appear on calculation of the available strategies $z \in Z, y \in Y, x \in X$ of player 1, player2 and player 3 respectively. Thus the operations $z \in Z, y \in Y, x \in X$ will be replaced by the following operations $z \in \mathcal{O}_{\varepsilon_z}(Z), y \in \mathcal{O}_{\varepsilon_y}(Y)$ and $x \in \mathcal{O}_{\varepsilon_x}(X)$, where the $\mathcal{O}_\varepsilon(\cdot)$ is the $\varepsilon-$neighborhood of sets. For simplicity we will used the following notations to denote the errors of mathematical operations: $E_x = (\varepsilon_x, \alpha_x)$, $E_y = (\varepsilon_y, \alpha_y)$, $E_z = (\varepsilon_z, \alpha_z)$. It should be mentioned that the errors $\alpha_x, \alpha_y, \alpha_z$ will diminish the maximum value of the payoff function for the players.

Using these errors the Stackelberg equilibrium profiles in the dynamic game of three players with three stages can be found by the following backward induction algorithm:

AA) for all $z \in \mathcal{O}_{\varepsilon_z}(Z)$ and $y \in \mathcal{O}_{\varepsilon_y}(Y)$ the payer 3 will choose the following best response function

$$x^*(z, y(z), E_x) \in R(H, z, y, E_x) = \left\{ x^* \in \mathcal{O}_{\varepsilon_x}(X) : \max_{x\in\mathcal{O}_{\varepsilon_x}(X)} H(z, y, x) - \alpha_x = H(z, y, x^*) \right\};$$

BB) for all $z \in \mathcal{O}_{\varepsilon_z}(Z)$ the payer 2 will choose the following best response function

$$y^*(z, E_y, E_x) \in L(F, z, E_y, E_x) =$$
$$= \left\{ y^*(z) \in \mathcal{O}_{\varepsilon_y}(Y) : \max_{y\in\mathcal{O}_{\varepsilon_y}(Y)} \min_{x^*(z,y(z))\in R(z,y,E_x)} F(z, y, x^*(z, y(z))) - \right.$$
$$\left. - \alpha_y = F(z, y^*(z, E_y, E_x), x^*(z, y^*(z, E_y, E_x), E_x)) \right\};$$

CC) the player 1 will choose the following best response reaction

$$z^*(E_z, E_y, E_x) \in Z^*(G, E_z, E_y, E_x) =$$
$$\left\{ z^* \in \mathcal{O}_{\varepsilon_z}(Z) : \max_{z\in\mathcal{O}_{\varepsilon_z}(Z)} \min_{y^*(z,E_y,E_x)\in L(z,E_y,E_x)} G(z, y^*(z, E_y, E_x) \right.$$
$$\left. - \alpha_z = G(z^*(E_z, E_y, E_x), y^*(z^*(E_z, E_y, E_x))) \right\}.$$

Let us introduce the following definition.

Definition 3 *For errors of mathematical operations* $E = (E_z, E_y, E_x)$ *the strategy profiles*

$$\{z^*(E_z, E_y, E_x), y^*(E_y, E_x), x^*(E_x)\} \equiv$$
$$\equiv \{z^*(E_z, E_y, E_x), y^*(z^*(E_z, E_y, E_x) E_y, E_x), x^*(z^*(E_z, E_y, E_x), y^*(z^*(E_z, E_y, E_x) E_y, E_x), E_x)\}$$

defined by the steps AA)-CC) of the backward induction algorithm are called the $E-$Stackelberg equilibrium profile in the dynamic game of three players with three stages Γ and are denoted by $SE(\Gamma; E_z, E_y, E_x)$.

Also, for simplicity we use the notations $E_z^l = (\varepsilon_z^l, \alpha_z^l)$, $E_y^l = (\varepsilon_y^l, \alpha_y^l)$, $E_x^r = (\varepsilon_x^r, \alpha_x^r)$ to denote the minimal errors of mathematical operations to determine the Stackelberg equilibrium profiles $SE(\Gamma; E_z^l, E_y^l, E_x^l)$ and $E_z^r = (\varepsilon_z^r, \alpha_z^r)$, $E_y^r = (\varepsilon_y^r, \alpha_y^r)$, $E_x^r = (\varepsilon_x^r, \alpha_x^r)$ to denote the maximal available errors of mathematical operations to determine the Stackelberg equilibrium profiles $SE(\Gamma; E_z^r, E_y^r, E_x^r)$. Mention that the errors $E, E^l \equiv \left(E_z^l, E_y^l, E_x^l\right), E^r \equiv \left(E_z^r, E_y^r, E_x^r\right)$ are inevitable for the computing process to obtain the Stackelberg equilibrium profile in the dynamic game of three players with three stages and can not be modified by the players.

So according to these errors we introduce the following definition.

Definition 4 *For errors $E_x^l = (\varepsilon_x^l, \alpha_x^l)$, $E_x^r = (\varepsilon_x^r, \alpha_x^r)$, $E_y^l = (\varepsilon_y^l, \alpha_y^l)$, $E_y^r = (\varepsilon_y^r, \alpha_y^r)$, $E_z^l = (\varepsilon_z^l, \alpha_z^l)$, $E_z^r = (\varepsilon_z^r, \alpha_z^r)$ the set of strategy profiles $\widetilde{SE}$ for which the following conditions*

$$SE\left(\Gamma; E_z^l, E_y^l, E_x^l\right) \subset \widetilde{SE} \subset SE\left(\Gamma; E_z^r, E_y^r, E_x^r\right)$$

are fulfilled is called the full set of $(E^l, E^r)-$ Stackelberg equilibrium profiles in the dynamic game of three players with three stages.

Let the $M = \{(G, F, H) / G : Z \times Y \to R, F : Z \times Y \times X \to R, H : Z \times Y \times X \to R\}$ denote the set of payoff functions which are used to describe the strategic form $\Gamma(M)$ of the dynamic game of three players with three stages. For any perturbations $\delta = (\delta_z, \delta_y, \delta_x)$ and $m = (G, F, H) \in M$ we introduce the set of $\delta-$ perturbed payoff functions m :

$$M_\delta(m) = \left\{ n = \left(G_{\delta_z}, F_{\delta_y}, H_{\delta_x}\right) : \sup_{z\in Z, y\in Y} \left|G(z,y) - G_{\delta_z}(z,y)\right| \le \delta_z, \right.$$

$$\left. \sup_{z\in Z, y\in Y, x\in X} \left|F(z,y,x) - F_{\delta_y}(z,y,x)\right| \le \delta_y, \sup_{z\in Z, y\in Y, x\in X} \left|H(z,y,x) - H_{\delta_x}(z,y,x)\right| \le \delta_x \right\}.$$

Thus for all $\delta-$perturbed payoff functions $\left(G_{\delta_z}, F_{\delta_y}, H_{\delta_x}\right) \in M_\delta(m)$ the strategic form of the dynamic game of three players with three stages

$$\Gamma_\delta = \left\langle Z, Y, X; G_{\delta_z} : Z \times Y \to R, F_{\delta_y} : Z \times Y \times X \to R, H_{\delta_x} : Z \times Y \times X \to R \right\rangle$$

is considered. This game is called *$\delta-$perturbed dynamic game of three players with three stages.*

We introduce the following definitions of stability of the $E-$Stackelberg equilibrium profile and the full $(E^l, E^r)-$ Stackelberg equilibrium profiles in the dynamic game of three players with three stages.

Definition 5 *The set of $E-$Stackelberg equilibrium profile $SE(\Gamma; E_z, E_y, E_x)$ is called stable on the sets of admissible errors E_z, E_y, E_x if there exists a value of perturbation $\delta > 0$, and such errors $\widetilde{E}_z, \widetilde{E}_y, \widetilde{E}_x$, that for all $\delta-$ perturbed functions $G_{\delta_z} \in M_\delta(m), F_{\delta_y} \in M_\delta(m)$ and $H_{\delta_x} \in M_\delta(m)$, which determine the strategic form game Γ_δ,the following conditions are satisfied*

1) $SE\left(\Gamma_\delta, \widetilde{E}_z, \widetilde{E}_y, \widetilde{E}_x, G_\delta(z,y)\right) \neq \varnothing$;

2) $\displaystyle\bigcup_{\substack{G_\delta \in M_\delta(m)\\ F_\delta \in M_\delta(m)\\ H_\delta \in M_\delta(m)}} SE\left(\Gamma_\delta, \widetilde{E}_z, \widetilde{E}_y, \widetilde{E}_x\right) \subset SE\left(\Gamma, E_z, E_y, E_x\right).$

Here $\widetilde{E} \equiv \left(\widetilde{E}_z,\widetilde{E}_y,\widetilde{E}_x\right)$ means the "errors of the players" to determine theirs strategies which forms the Stackelberg equilibrium profiles. Then the set of parameters $\left\{\delta, \widetilde{E}_z, \widetilde{E}_y, \widetilde{E}_x\right\}$ for which the conditions 1)-2) of the definition 5 are satisfied, is called *set of stability of $E-$Stackelberg equilibrium profile* in the dynamic game of three players with three stages and is denoted by $\mathcal{S}(\delta, E_z, E_y, E_x)$.

So, for all $\left\{\delta, \widetilde{E}_z, \widetilde{E}_y, \widetilde{E}_x\right\} \in \mathcal{S}(\delta, E_z, E_y, E_x)$, the $\widetilde{E}-$Stackelberg equilibrium profile in the $\delta-$perturbed dynamic game Γ_δ is the $E-$Srackelberg equilibrium profile in the nonperturbed dynamic game Γ. On other hand, if for all parameters $\left\{\delta, \widetilde{E}_z, \widetilde{E}_y, \widetilde{E}_x\right\} \in \mathcal{S}(\delta, E_z, E_y, E_x)$ the strategy profile (z^*, y^*, x^*) belongs to $SE\left(\Gamma_\delta, \widetilde{E}_z, \widetilde{E}_y, \widetilde{E}_x\right)$, it is true that $(z^*, y^*, x^*) \in SE(\Gamma, E_z, E_y, E_x)$.

Also, the stability analysis of the $E-$Stackelberg equilibrium profiles in the dynamic game of three players with three stages consists of determining the stability set $\mathcal{S}(\delta, E_z, E_y, E_x)$.

Definition 6 *The full set of $(E^l, E^r)-$Stackelberg equilibrium profile $\widetilde{SE}$ is called stable on the sets of admissible minimal and maximal errors $E^r_z, E^r_y, E^r_x, E^l_z, E^l_y, E^l_x$, if there exists a value of perturbation $\delta > 0$, and such errors $\widetilde{E}^r_z$, $\widetilde{E}^r_y$, $\widetilde{E}^r_x$, $\widetilde{E}^l_z$, $\widetilde{E}^l_y$, $\widetilde{E}^l_x$, that for all $\delta-$ perturbed functions $G_{\delta_z} \in M_\delta(m), F_{\delta_y} \in M_\delta(m)$ and $H_{\delta_x} \in M_\delta(m)$, which determine the strategic form game Γ_δ, the following conditions are satisfied*

1) $SE\left(\Gamma_\delta, \widetilde{E}^r_z, \widetilde{E}^r_y, \widetilde{E}^r_x\right) \neq \emptyset, \quad SE\left(\Gamma_\delta, \widetilde{E}^l_z, \widetilde{E}^l_y, \widetilde{E}^l_x\right) \neq \emptyset;$

2) $SE\left(\Gamma_\delta, \widetilde{E}^l_z, \widetilde{E}^l_y, \widetilde{E}^l_x\right) \subset SE\left(\Gamma_\delta, \widetilde{E}^r_z, \widetilde{E}^r_y, \widetilde{E}^r_x\right);$

3) $$\bigcup_{\substack{G_{\delta_z} \in M_\delta(m)\\ F_{\delta_y} \in M_\delta(m)\\ H_{\delta_x} \in M_\delta(m)}} SE\left(\Gamma_\delta, \widetilde{E}^r_z, \widetilde{E}^r_y, \widetilde{E}^r_x\right) \subset SE\left(\Gamma, E^r_z, E^r_y, E^r_x\right);$$

4) $$\bigcap_{\substack{G_{\delta_z} \in M_\delta(m)\\ F_{\delta_y} \in M_\delta(m)\\ H_{\delta_x} \in M_\delta(m)}} SE\left(\Gamma_\delta, \widetilde{E}^l_z, \widetilde{E}^l_y, \widetilde{E}^l_x\right) \supset SE\left(\Gamma, E^l_z, E^l_y, E^l_x\right);$$

Here $\widetilde{E}^l \equiv \left(\widetilde{E}^l_z, \widetilde{E}^l_y, \widetilde{E}^l_x\right), \widetilde{E}^r \equiv \left(\widetilde{E}^r_y, \widetilde{E}^r_x, \widetilde{E}^r_z\right)$ mean the *minimal and respectively maximal errors of the players* to determine their strategies which forms the Stackelberg equilibrium profiles. Then the set of parameters $\left\{\delta, \widetilde{E}^r_z, \widetilde{E}^r_y, \widetilde{E}^r_x, \widetilde{E}^l_z, \widetilde{E}^l_y, \widetilde{E}^l_x\right\}$ for which the conditions 1)-4) of the definition 6 are satisfied is called *full set of stability of $(E^l, E^r)-$Stackelberg equilibrium profile* in the dynamic game of three players with three stages and is denoted $\mathcal{S}\left(\delta, E^l_z, E^l_y, E^l_x, E^r_z, E^r_y, E^r_x\right)$.

So, for all $\left\{\delta, \widetilde{E}^r_z, \widetilde{E}^r_y, \widetilde{E}^r_x, \widetilde{E}^l_z, \widetilde{E}^l_y, \widetilde{E}^l_x\right\} \in \mathcal{S}\left(\delta, E^l_z, E^l_y, E^l_x, E^r_z, E^r_y, E^r_x\right)$, the full set of $\left(\widetilde{E}^l, \widetilde{E}^r\right)-$Stackelberg equilibrium profile in the $\delta-$ perturbed dynamic game Γ_δ is the full set of $\left(E^l, E^r\right)-$Srackelberg equilibrium profile in the nonperturbed dynamic game Γ. On other hand, if for all parameters $\left\{\delta, \widetilde{E}^l_z, \widetilde{E}^l_y, \widetilde{E}^l_x, \widetilde{E}^r_z, \widetilde{E}^r_y, \widetilde{E}^r_x\right\} \in \mathcal{S}\left(\delta, E^l_z, E^l_y, E^l_x, E^r_z, E^r_y, E^r_x\right)$ the set of equilibrium profiles $\widehat{SE}$ satisfies the conditions $SE\left(\Gamma_\delta, \widetilde{E}^l_z, \widetilde{E}^l_y, \widetilde{E}^l_x\right) \subset \widehat{SE} \subset SE\left(\Gamma_\delta, \widetilde{E}^r_z, \widetilde{E}^r_y, \widetilde{E}^r_x\right)$, it also satisfies the following $SE\left(\Gamma, E^l_z, E^l_y, E^l_x\right) \subset \widehat{SE} \subset SE\left(\Gamma, E^r_z, E^r_y, E^r_x\right)$.

The stability analysis of the full set of $\left(E^l, E^r\right)-$ Stackelberg equilibrium profiles in the dynamic game of three players with three stages consists of determining the stability set $S\left(\delta, E^l_z, E^l_y, E^l_x, E^r_z, E^r_y, E^r_x\right)$.

We can prove the following theorem which describes the stability set $S\left(\delta, E^l_z, E^l_y, E^l_x, E^r_z, E^r_y, E^r_x\right)$.

Theorem 7 *If for the dynamic game of three players with three stages in strategic form Γ the conditions of the theorem 2 are satisfied then the stability set to determine the full set of $\left(E^l, E^r\right)-$ Stackelberg equilibrium profiles is*

$$\mathcal{S}\left(\delta, E^l_z, E^l_y, E^l_x, E^r_z, E^r_y, E^r_x\right) =$$
$$= \Big\{ \widetilde{E}^r_z, \widetilde{E}^r_y, \widetilde{E}^r_x, \widetilde{E}^l_z, \widetilde{E}^l_y, \widetilde{E}^l_x \Big/ \widetilde{\varepsilon}^r_z \leq \varepsilon^r_z - \delta_z, \widetilde{\alpha}^r_z \leq \alpha^r_z - 2\delta_z, \widetilde{\varepsilon}^l_z \geq \varepsilon^l_z + \delta_z, \widetilde{\alpha}^l_z \geq \alpha^l_z + 2\delta_z,$$
$$\widetilde{\varepsilon}^r_y \geq \varepsilon^r_y + \delta_y, \widetilde{\alpha}^r_y \geq \alpha^r_y + 2\delta_y, \widetilde{\varepsilon}^l_y \leq \varepsilon^l_y - \delta_y, \widetilde{\alpha}^l_y \leq \alpha^l_y - 2\delta_y,$$
$$\widetilde{\varepsilon}^r_x \leq \varepsilon^r_x - \delta_x, \widetilde{\alpha}^r_x \leq \alpha^r_x - 2\delta_x, \widetilde{\varepsilon}^l_x \geq \varepsilon^l_x + \delta_x, \widetilde{\alpha}^l_x \geq \alpha^l_x + 2\delta_x \Big\}.$$

Proof. Let $\mathcal{E} = (E_z, E_y, E_x)$, $\mathcal{E}^r = \left(E^r_z, E^r_y, E^r_x\right)$, $\mathcal{E}^l = \left(E^l_z, E^l_y, E^l_x\right)$. The theorem 2 implies that $SE\left(\Gamma_\delta, \widetilde{E}^r_z, \widetilde{E}^r_y, \widetilde{E}^r_x\right) \neq \emptyset$, $SE\left(\Gamma_\delta, \widetilde{E}^l_z, \widetilde{E}^l_y, \widetilde{E}^l_x\right) \neq \emptyset$. As errors $\widetilde{\mathcal{E}}^l$ give the better precision of the solution than errors $\widetilde{\mathcal{E}}^r$ we have the condition 2) of the definition 6.

Let $\left(z^*\left(\widetilde{\mathcal{E}}^r\right), y^*\left(\widetilde{\mathcal{E}}^r\right), x^*\left(\widetilde{\mathcal{E}}^r\right)\right) \in SE\left(\Gamma_\delta, \widetilde{\mathcal{E}}^r\right)$ for arbitrary perturbed functions $\left(G_{\delta_z}, F_{\delta_y}, H_{\delta_x}\right) \in M_\delta(G, F, H)$, i. e. the following conditions

$$\begin{cases} x^*\left(\widetilde{\mathcal{E}}^r\right) \in R\left(H_{\delta_x}, z^*\left(\widetilde{\mathcal{E}}^r\right), y^*\left(\widetilde{\mathcal{E}}^r\right)\right), \\ y^*\left(\widetilde{\mathcal{E}}^r\right) \in L\left(F_{\delta_y}, z^*\left(\widetilde{\mathcal{E}}^r\right)\right), \\ z^*\left(\widetilde{\mathcal{E}}^r\right) \in Z^*\left(G_{\delta_z}, \widetilde{\mathcal{E}}^r\right) \end{cases}$$

are fulfilled. Then for errors $\widetilde{\varepsilon}^r_z \leq \varepsilon^r_z - \delta_z$, $\widetilde{\varepsilon}^r_y \geq \varepsilon^r_y + \delta_y$, $\widetilde{\varepsilon}^r_x \leq \varepsilon^r_x - \delta_x$, $\widetilde{\alpha}^r_z \leq \alpha^r_z - 2\delta_z$, $\widetilde{\alpha}^r_y \geq \alpha^r_y + 2\delta_y$, and $\widetilde{\alpha}^r_x \leq \alpha^r_x - 2\delta_x$ it is easy to prove that

$$\begin{cases} x^*\left(\widetilde{\mathcal{E}}^r\right) \in R\left(H, z^*\left(\mathcal{E}^r\right), y^*\left(\mathcal{E}^r\right)\right), \\ y^*\left(\widetilde{\mathcal{E}}^r\right) \in L\left(F, z^*\left(\mathcal{E}^r\right)\right), \\ z^*\left(\widetilde{\mathcal{E}}^r\right) \in Z^*\left(G, \mathcal{E}^r\right) \end{cases}$$

which imply $\left(z^*\left(\widetilde{\mathcal{E}}^r\right), y^*\left(\widetilde{\mathcal{E}}^r\right), x^*\left(\widetilde{\mathcal{E}}^r\right)\right) \in SE\left(\Gamma, \mathcal{E}^r\right)$. So, the condition 3) of the definition 6 is obtained.

Let $\left(z^*\left(\mathcal{E}^l\right), y^*\left(\mathcal{E}^l\right), x^*\left(\mathcal{E}^l\right)\right) \in SE\left(\Gamma, \mathcal{E}^l\right)$, so the following conditions are fulfilled:

$$\begin{cases} x^*\left(\mathcal{E}^l\right) \in R\left(H, z^*\left(\mathcal{E}^l\right), y^*\left(\mathcal{E}^l\right)\right), \\ y^*\left(\mathcal{E}^l\right) \in L\left(F, z^*\left(\mathcal{E}^l\right)\right), \\ z^*\left(\mathcal{E}^l\right) \in Z^*\left(G, \mathcal{E}^l\right). \end{cases}$$

For all arbitrary perturbed functions $\left(G_{\delta_z}, F_{\delta_y}, H_{\delta_x}\right) \in M_\delta(G, F, H)$ and errors $\widetilde{\varepsilon}^l_z \geq \varepsilon^l_z + \delta_z$, $\widetilde{\varepsilon}^l_y \leq \varepsilon^l_y - \delta_y$, $\widetilde{\varepsilon}^l_x \geq \varepsilon^l_x + \delta_x$, $\widetilde{\alpha}^l_z \geq \alpha^l_z + 2\delta_z$, $\widetilde{\alpha}^l_y \leq \alpha^l_y - 2\delta_y$, $\widetilde{\alpha}^l_x \geq \alpha^l_x + 2\delta_x$ it is easy to prove that

$$\begin{cases} x^*\left(\widetilde{\mathcal{E}}^l\right) \in R\left(H_{\delta_x}, z^*\left(\widetilde{\mathcal{E}}^l\right), y^*\left(\widetilde{\mathcal{E}}^l\right)\right), \\ y^*\left(\widetilde{\mathcal{E}}^l\right) \in L\left(F_{\delta_y}, z^*\left(\widetilde{\mathcal{E}}^l\right)\right), \\ z^*\left(\widetilde{\mathcal{E}}^l\right) \in Z^*\left(G_{\delta_z}, \widetilde{\mathcal{E}}^l\right) \end{cases}$$

which imply $\left(z^*\left(\mathcal{E}^l\right), y^*\left(\mathcal{E}^l\right), x^*\left(\mathcal{E}^l\right)\right) \in SE\left(\Gamma_\delta, \widetilde{\mathcal{E}}^l\right)$. So the condition 4) of the definition 6 is verified. The theorem is proved.

Due to the theorem 7 for perturbation δ_z of the function G and for maximal error ε_z^r of the operation "$z \in \mathcal{O}_{\varepsilon_z^r}(Z)$" and minimal error ε_z^l of the operation "$z \in \mathcal{O}_{\varepsilon_z^l}(Z)$", the maximal $\widetilde{\varepsilon}_z^r$ and respectively minimal $\widetilde{\varepsilon}_z^l$ errors of the player 1 to execute the operations "$z \in \mathcal{O}_{\widetilde{\varepsilon}_z^r}(Z)$" and respectively "$z \in \mathcal{O}_{\widetilde{\varepsilon}_z^l}(Z)$" satisfy the conditions $\widetilde{\varepsilon}_z^r \leq \varepsilon_z^r - \delta_z$ and $\widetilde{\varepsilon}_z^l \geq \varepsilon_z^l + \delta_z$. Similarly, for the maximal error α_z^r of the operation "$z^* \in Z^*\left(G, E_z^r, E_y^r, E_x^r\right)$" and the minimal error α_z^l of the operation "$z^* \in Z^*\left(G, E_z^l, E_y^l, E_x^l\right)$", the maximal $\widetilde{\alpha}_z^r$ and minimal $\widetilde{\alpha}_z^l$ errors of the player 1 to execute the operations "$z^* \in Z^*\left(G_{\delta_z}, \widetilde{E}_z^r, \widetilde{E}_y^r, \widetilde{E}_x^r\right)$"and "$z^* \in Z^*\left(G_{\delta_z}, \widetilde{E}_z^l, \widetilde{E}_y^l, \widetilde{E}_x^l\right)$" respectively, satisfy the conditions $\widetilde{\alpha}_z^r \leq \alpha_z^r - 2\delta_z$ and $\widetilde{\alpha}_z^l \leq \alpha_z^l + 2\delta_z$.

Also, for perturbation δ_y of the function F, maximal error ε_y^r of the operation "$y \in \mathcal{O}_{\varepsilon_y^r}(Y)$" and minimal error ε_y^l for the operation "$y \in \mathcal{O}_{\varepsilon_y^l}(Y)$", the maximal $\widetilde{\varepsilon}_y^r$ and minimal $\widetilde{\varepsilon}_y^l$ errors of the player 2 to execute the operations "$y \in \mathcal{O}_{\widetilde{\varepsilon}_y^r}(Y)$" and respectively "$y \in \mathcal{O}_{\widetilde{\varepsilon}_y^l}(Y)$" satisfy the conditions $\widetilde{\varepsilon}_y^r \geq \varepsilon_y^r + \delta_y$ and $\widetilde{\varepsilon}_y^l \leq \varepsilon_y^l - \delta_y$. Similarly, for the maximal error α_y^r of the operation "$y^* \in L(F, z, E_y^r, E_x^r)$", and the minimal error α_y^l of the operation "$y^* \in L(F, z, E_y^l, E_x^l)$", the maximal $\widetilde{\alpha}_y^r$ and respectively minimal $\widetilde{\alpha}_y^l$ errors of the player 2 to execute the operation "$y^* \in L(F_{\delta_y}, \widetilde{E}_y^r, \widetilde{E}_x^r)$" and the operation "$y^* \in L(F_{\delta_y}, \widetilde{E}_y^l, \widetilde{E}_x^l)$" satisfy the conditions $\widetilde{\alpha}_y^r \geq \alpha_y^r + 2\delta_y$ and $\widetilde{\alpha}_y^l \leq \alpha_y^l - 2\delta_y$.

In the same way, for perturbation δ_x of the function H, maximal error ε_x^r of the operation "$x \in \mathcal{O}_{\varepsilon_x^r}(X)$" and minimal error ε_x^l for the operation "$x \in \mathcal{O}_{\varepsilon_x^l}(X)$", the maximal $\widetilde{\varepsilon}_x^r$ and respectively minimal $\widetilde{\varepsilon}_x^l$ errors of the player 3 to execute the operations "$x \in \mathcal{O}_{\widetilde{\varepsilon}_x^r}(X)$" and respectively "$x \in \mathcal{O}_{\widetilde{\varepsilon}_x^l}(X)$" satisfy the conditions $\widetilde{\varepsilon}_x^r \leq \varepsilon_x^r - \delta_x$ and $\widetilde{\varepsilon}_x^l \geq \varepsilon_x^l + \delta_x$. For the maximal error α_x^r of the operation "$x^* \in R(H, z, y, E_x^r)$" and the minimal error α_x^l of the operation "$x^* \in R(H, z, y, E_x^l)$" the maximal $\widetilde{\alpha}_x^r$ and minimal $\widetilde{\alpha}_x^l$errors of the player 3 to execute the operation "$x^* \in R(H_{\delta_x}, z, y, \widetilde{E}_x^r)$" and the operation "$x^* \in R(H_{\delta_x}, z, y, \widetilde{E}_x^l)$" respectively satisfy the conditions $\widetilde{\alpha}_x^r \leq \alpha_x^r - 2\delta_x$ and $\widetilde{\alpha}_x^l \geq \alpha_x^l + 2\delta_x$.

The qualitative results of these theorem are the following. For ensure the stability of the full set of $\left(E^l, E^r\right) -$ Stackelberg equilibrium profiles in the dynamic game of three players with three stage it is sufficient that the player 1 decreases by δ_z the errors $\widetilde{\varepsilon}_z^r$ relative to error ε_z^r, and by $2\delta_z$ the error $\widetilde{\alpha}_z^r$ relative to error α_z^r; respectively he increases by δ_z the errors $\widetilde{\varepsilon}_z^l$ relative to error ε_z^l, and by $2\delta_z$ the error $\widetilde{\alpha}_z^l$ relative to error α_z^l. For the player 2 the operations are reverse: it is sufficient to increase by δ_y the errors $\widetilde{\varepsilon}_y^r$ relative to error ε_y^r, and by $2\delta_y$ the error $\widetilde{\alpha}_y^r$ relative to error α_y^r and respectively to decrease by δ_y the errors $\widetilde{\varepsilon}_y^l$ relative to error ε_y^l, and by $2\delta_y$ the error $\widetilde{\alpha}_y^l$ relative to error α_y^l. By the player 3 it is similar to the player 1: he decreases by δ_x the errors $\widetilde{\varepsilon}_x^r$ relative to error ε_x^r, and by $2\delta_x$ the error $\widetilde{\alpha}_x^r$ relative to error α_x^r; respectively he increases by δ_x the errors $\widetilde{\varepsilon}_x^l$ relative to error ε_x^l, and by $2\delta_x$ the error $\widetilde{\alpha}_x^l$ relative to error α_x^l.

The stable set $\mathcal{S}(\delta, E_z, E_y, E_x)$ is described by the following corollary.

Corollary 8 *If the dynamic game of three players with three stages in strategic form Γ is satisfied the conditions of the theorem 2 then the stability set $\mathcal{S}(\delta, E_z, E_y, E_x)$ for determine the $E-$Stackelberg equilibrium profiles in the dynamic game of three players with three stage is satisfied the following conditions:*

$$\begin{aligned}
\widetilde{\varepsilon}_z &\leq \varepsilon_z - \delta_z; \widetilde{\alpha}_z \leq \alpha_z^r - 2\delta_z;\\
\widetilde{\varepsilon}_y &\geq \varepsilon_y + \delta_y; \widetilde{\alpha}_y \geq \alpha_y + 2\delta_y;\\
\widetilde{\varepsilon}_x &\leq \varepsilon_x - \delta_x; \widetilde{\alpha}_x \leq \alpha_x - 2\delta_x.
\end{aligned}$$

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