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OPTIMUM DESIGNING OF ANISOTROPIC ROUND PLATES WITH RESPECT TO EIGENVALUES

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Se cercetează problema de proiectare a plăcilor anizotropice rotunde fixate pe contur cu valoarea maximală (minimală) a frecvenţei transversale libere. Au fost examinate două cazuri de fixare: placă simplu rezemată şi placă rezemată rigid pe contur. Optimizarea a fost realizată folosind metodele numerice.

The problem of designing fibre-reinforced round fixed plates with maximum (minimum) fundamental frequency of free transverse vibrations is considered. Two aspects of fixing are considered: simply supported plate and rigidly clamped plate. Optimization was performed using numerical approach.

Problem definition

Let us consider a round fixed anisotropic plate of radius R, whose fixing does not break symmetry of the plate. We refer the plate to a polar coordinate system r,θ whose origin coincides with the centre of the plate. Bending strains of the plate can be described by function of displacement $w(r,\theta,t)$. For simple harmonic vibration we have $w(r,\theta,t)=w^0(r,\theta)e^{i\varpi t}$, where $w^0(r,\theta)$ amplitude values of displacement (later the index 0 will be omitted). Anisotropic properties will be described by distribution of angle $\varphi(r)$ between the first anisotropy axis and axis r of coordinate system.

For radial-symmetrical distributions of angle $\varphi(r)$ the problem about free oscillations of the round plate can be reduced to the one-dimensional. In particular, for rotationally symmetric oscillation the displacement of the plate is described by rotationally symmetric function $w(r)$. Components of a strain tensor in polar coordinates are given by expressions [1]:

$$w_{rr}=\frac{d^2w}{dr^2},\quad w_{r\theta}=0,\quad w_{\theta\theta}=\frac{1}{r}\frac{dw}{dr} \tag{1}$$

According to a principle of the Rayleigh fundamental frequency can be written in the form [2]:

$$\omega^2(\varphi)=\min_{w\in B}\frac{U}{T}, \tag{2}$$

where the potential energy of a strain is given by relation

$$U=\frac{1}{2}2\pi\int_0^R(D_{11}(\varphi)w_{rr}^2+2D_{12}(\varphi)w_{rr}\frac{w_r}{r}+D_{22}(\varphi)\frac{w_r^2}{r^2})rdr, \tag{3}$$

$D_{11}(\varphi),\ D_{12}(\varphi),\ D_{22}(\varphi)$ - coefficients of an anisotropy, which are dependent on required function $\varphi(r)$:

$$\begin{aligned} D_{11}(\varphi)&=D_1\cos^4\varphi+2D_3\sin^2\varphi\cos^2\varphi+D_2\sin^4\varphi\\ D_{22}(\varphi)&=D_1\sin^4\varphi+2D_3\sin^2\varphi\cos^2\varphi+D_2\cos^4\varphi,\\ D_{12}(\varphi)&=D_3+(D_1+D_2-2D_3)\sin^2\varphi\cos^2\varphi \end{aligned} \tag{4}$$

where D_1,D_2,D_3 are constants which can be expressed in terms of orthotropy constants as:

$$D_1=\frac{E_1}{1-\nu_{12}\nu_{21}},\ D_2=\frac{E_2}{1-\nu_{12}\nu_{21}},\ D_3=E_{12}\nu_{12}+2G_{12} \tag{5}$$

The kinetic energy of an oscillating plate is determined as follows [3]:

$$T=\frac{1}{2}\rho h2\pi\int_o^R w^2(r)rdr, \tag{6}$$

where ρ,h - a denseness and width of the plate.

From a necessary condition of minimum of the functional (2) $\delta\omega^2(\varphi)=0$, we will obtain the equation of the free rotationally symmetric oscillations of a round plate and boundary conditions for function of displacement:

$$(rD_{11}w_{rr})_{rr}-(D_{12}w_{rr})_r+(D_{12}w_r)_{rr}-\left(\frac{D_{22}}{r}w_r\right)_r=\rho h\omega^2 rw \tag{7}$$

At centre of a plate for all requirements of fixing displacement obeys

$$w_r(0)=0,\quad (rD_{11}w_{rr})_r+\frac{dD_{12}}{dr}w_r-D_{22}\frac{w_r}{r}=0 \tag{8}$$

for rigidly clamped plate when $r=R$:

$$w(R)=0,\ w_r(R)=0 \tag{9}$$

Boundary conditions for simply supported plate are :

$$w(R)=0,\quad D_{11}w_{rr}(R)+D_{12}\frac{w_r(R)}{R}=0 \tag{10}$$

The problem of optimization is formulated as follows: to find distribution of anisotropy angle $\varphi(r)$ for which fundamental frequency (2) obtains maximum (minimum) value:

$$\varphi^*(r):\ \ \omega^2(\varphi)\to \max_{\varphi}(\min_{\varphi}) \tag{11}$$

Optimum distribution of anisotropy angle $\varphi^*(r)$ was seeking in the class of rotationally symmetric functions in the assumption, that the fundamental oscillation frequency is simple and the eigen-function is rotationally symmetric.

Discretization of a problem of optimum designing.

The problem formulated above was solved numerically. Spectrum of free oscillations was found using the variation-difference method. According to this method the range of integration was divided into n intervals of equal length by points $r_i=ih,\ \ h=R/n,\ \ i=\overline{0,n}$. Integral expressions for potential and a kinetic energy were approximated using quadrature formula of trapezoid:

$$U=\sum_{i=0}^{n}a_i f_i \qquad T=\sum_{i=0}^{n}a_i r_i w_i^2 \tag{12}$$

where $a_0=a_n=\frac{h}{2},\ a_i=h,\ i=\overline{1,n-1}$ and by f_i is denoted the denseness of a potential energy in the point r_i:

$$f_i=(D_{11}(\varphi_i)w_{rr}^2(r_i)+2D_{12}(\varphi_i)w_{rr}(r_i)\frac{w_r(r_i)}{r_i}+D_{22}(\varphi_i)\frac{w_r^2(r_i)}{r_i^2})r_i, \tag{13}$$

i=1,2, … n

where φ_i - a value of distribution of anisotropy angle in the point r_i.

Derivatives in the given expression were substituted by finite-difference approximations of the second order of accuracy. As a result following approximate expressions for eigenvalues has been obtained

$$\omega^2(\vec\varphi)=\min_{\vec w}\frac{\sum_{i=0}^{n}r_i a_i\{\vec\delta_i^t\}[k^i\varphi_i)]\{\vec\delta_i\}}{\sum_{i=0}r_i a_i w_i^2},\quad \{\vec\delta_i^t\}=(w_{i-1},w_i,w_{i+1}), \tag{14}$$

Where $\vec w=(w_0,w_1,....w_n)$ - is the vector of displacements, $\vec\varphi=(\varphi_1,\varphi_2,..\varphi_n)$ is the vector of distributions of anisotropy angle, $[k^i(\varphi_i)]$ - the symmetrical matrix, which elements can be calculated using expressions:

$$k_{11}^i = \frac{D_{11}^i}{h^4} - \frac{D_{12}^i}{h^3 r_i} + \frac{D_{22}^i}{4h^2 r_i^2},$$

$$k_{12}^i = -\frac{2D_{11}^i}{h^4} + \frac{D_{12}^i}{h^3 r_i}, \quad k_{13}^i = \frac{D_{11}^i}{h^4} - \frac{D_{22}^i}{4h^2 r_i^2} \tag{15}$$

$$k_{22}^i = \frac{4D_{11}^i}{h^4},\ k_{23}^i = -\frac{2D_{11}^i}{h^4} - \frac{D_{12}^i}{h^3 r_i}, \quad k_{33}^i = \frac{D_{11}^i}{h^4} + \frac{D_{12}^i}{h^3 r_i} + \frac{D_{22}^i}{4h^2 r_i^2}$$

Let's write expression (14) as

$$\omega^2(\vec{\varphi}) = \min_{\vec{w}} \frac{\vec{w}^t [K] \vec{w}}{\vec{w}^t [M] \vec{w}} \tag{16}$$

Where [K] - a symmetrical matrix obtained as a result of assembling matrixes $[k^i]$, [M] a diagonal matrix with elements $[M]_{ii} = r_i a_i$.

By writing a condition of minimum for $\omega^2(\vec{\varphi})$: $\frac{\partial \omega^2}{\partial w_i} = 0,\ i = \overline{0,n}$, we'll obtain algebraic problem on eigenvalues

$$[K]\vec{w} = \omega^2 [M]\vec{w} \tag{17}$$

The problem of optimization will consist of computing the vector $\vec{\varphi} = \{\varphi_i\},\ i = 0,n$ which supplies an extremum to the basic eigenvalue (17).

The method of sequential optimization has been applied to solve this problem. According to this method were defined the improving variations of fundamental frequency:

$$\delta\vec{\varphi} = \tau\vec{\Lambda}(\vec{\varphi}, \vec{w}) \tag{18}$$

For simple eigenvalues gradient was calculated by formula

$$\vec{\Lambda} = \{\Lambda_0, \Lambda_1, \quad \Lambda_n\}, \quad \Lambda_i = \vec{\delta}_i^t \frac{\partial [k^i]}{\partial \varphi_i} \vec{\delta}_i, \quad i = \overline{0,n}, \tag{19}$$

And the new approximation was given by:

$$\vec{\varphi}^{n+1} = \vec{\varphi}^n + \tau\vec{\Lambda}(\vec{\varphi}^n, \vec{w}^n) \tag{20}$$

Numerical solution of the problem of optimum designing

Numerical calculations were performed for locally-orthotropic round plates of radius r=1 with coefficients of ortothotropy $E_1 = 1,\ E_2 = 1/3,\ G_{12} = 1/6,\ \nu_{12} = 0{,}08,\ \nu_{21} = 0{,}25$. The radius of the plate was divided into 100 segments. For analysis of dependence of fundamental frequency on distribution of anisotropy angle were performed calculations for a case of constant distribution of anisotropy angle. The calculations have displayed considerable sensitivity of fundamental frequency to modifications of parameters of optimization. Below in the table 1 are given the values of quadrates of fundamental frequency for some distribution of anisotropy angle for simply supported plate.

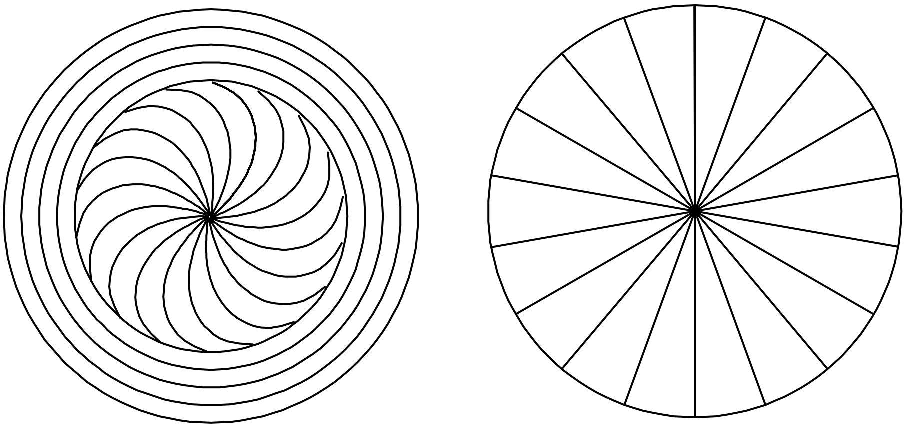
Table 1

Eigenvalues of the free transverse vibrations anisotropic round simply supported plate

Angle of an anisotropy $\varphi(r)$	Quadrate of fundamental frequency ω^2
π/2	1,35864
π/3	1,26290
π/4	1,14346
π/6	1,00854
0	0,87076

For the optimum solution the value of quadrate of fundamental frequency is 1,38175.

Fibre orientations for simply supported plate are given on fig 1. On the left is presented fibre orientation with maximum value of fundamental frequency. We can see two different zones of reinforcement. First zone is characterized by constant fibre orientation $\varphi(r) = \pi / 2$. The transition from this zone to the second zone takes place at *r*=0.7. On the right is given fibre orientation that minimizes fundamental frequency. In this case fibers are oriented along radius of the plate $\varphi = 0$.



1 2

Fig.1. Optimum fiber orientations for simply supported round plate.

Below in table 2 are given results of parametric calculations for rigidly clamped plate.

Table 2

Eigenvalues of the free transverse vibrations of anisotropic round rigidly clamped plate

$\varphi(r)$	ω^2
$\pi/2$	4,28719
$\pi/3$	4,24314
$\pi/4$	4,6296
$\pi/6$	5,59858
0	6,5857

For the optimum solution maximal value of quadrate of fundamental frequency $\omega^2 = 7{,}6655$. The optimum fiber orientation is presented on fig. 2 – first image. Calculations were also performed to find fiber orientation with a minimum value of fundamental frequency. The solution is presented on fig. 2 - 2 image. The value of frequency quadrate is $\omega^2 = 3{,}3572$.

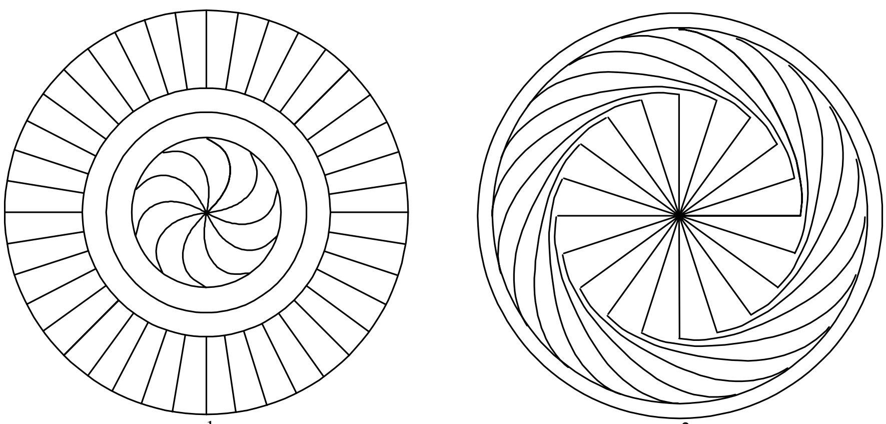


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Fig.2. Optimum fiber orientations for rigidly clamped plate.

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