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ABOUT RATES OF CONVERGENCE IN THE LIMIT THEOREM

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În lucrare se propune o estimare a ratei de convergență a densității sumei de variabile aleatoare în raport cu densitatea probabilității de repartiție. Studii similare au fost făcute şi de către alți matematicieni. Aici putem menționa lucrarea lui W. Macht şi Wolf W. *„On the local central limit theorem"* [2], în care aproximarea densității probabilității unui şir de variabile aleatoare este făcută cu referire la densitatea probabilității distribuției normale standarde:

$$\sup_x \left| p_{Z_n(x)} - \phi(x) \right| = O\left(\frac{\sqrt{\ln \Delta_n}}{\Delta_n} \right)^{\gamma}.$$

Principalul rezultat al acestui articol se conține în teorema 2.1, care stabileşte rata de convergență a densității probabilității unui şir de variabile aleatoare în raport cu densitatea probabilității de repartiție a variantei cu una şi cu trei grade de libertate.

1. Introduction

We consider a sequence of random variables (Y_n), where $Y_n = \dfrac{\sum_{i=1}^{n} X_i}{\operatorname{var} \sum_{i=1}^{n} X_i}$, (X_i) represent a sequence of positive independent random variables having the moment of three finite, with the probability density (ρ_{Y_n}) and the characteristic function (φ_{Y_n}).

Let (Δ_n) be a sequence of real positive numbers, with an infinite limit if $n \to \infty$.

The following definition is required:

Definition 1.1. A continuous function $q : R \to R$ is said to be Hölder continuous by exponent $\gamma \in (0,1]$ if

$$\sup_x |q(x+y) - q(x)| = O\left(|y|^{\gamma}\right) \text{when } y \to 0.$$

If $\gamma = 1$, function q is Lipschitz-continuous.

It is known that a sequence of functions $(q_n(x))$ is Hölder continuous if the constant from O symbol does not depend on n.

2. The rate of convergence in the local limit theorem

For the beginning we introduce a lemma that approximates the probability density $(\rho_{Y_n}(x))$ towards the density $(\rho_{Y_n}(x,T))$ defined as follows:

$$\rho_{Y_n}(x,T) = \int_{-\infty}^{+\infty} \rho_{Y_n}\left(x - \frac{y}{T} \right) f_{Y_n^*}(y)dy, \ T > 0,$$

where $(f_{Y_n^*}(x))$ is the probability density of the random variable (Y_n^*).

Lemma 2.1. Let $(\rho_{Y_n}(x))$ be it the bounded probability density. Then, for $\alpha > 1$ and $T \to \infty$ we have

$$\left| \rho_{Y_n}(x,T) - \rho_{Y_n}(x) \right| = \int_{|y| \le T^{\alpha}} \left| \rho_{Y_n}\left(x - \frac{y}{T} \right) - \rho_{Y_n}(x) \right| f_{Y_n^*}(y)dy + O(1 - T).$$

Proof. According to $(\rho_{Y_n}(x,T))$ definition for the difference $\rho_{Y_n}(x,T)-\rho_{Y_n}(x)$ we give a superior bounded:

$$\left|\rho_{Y_n}(x,T)-\rho_{Y_n}(x)\right| = \int_{|y|\le T^\alpha}\left|\rho_{Y_n}\left(x-\frac{y}{T}\right)-\rho_{Y_n}(x)\right| f_{Y_n^*}(y)dy + \int_{|y|>T^\alpha}\left|\rho_{Y_n}\left(x-\frac{y}{T}\right)\right| f_{Y_n^*}(y)dy + \sup_x \rho_{Y_n}(x)\int_{|y|>T^\alpha} f_{Y_n^*}(y)dy \overset{\text{not.}}{=} I + J$$

It is obvious that $I = O(1-T)$ and $J = O(1-T)$ if $T \to \infty$.

It's clear that, if $\left|f_{Y_n^*}(y)\right| \le 1$, forall $y \ge 0$, we have:

$$I = \int_{|y|>T^\alpha}\left|\rho_{Y_n}\left(x-\frac{y}{T}\right)\right| f_{Y_n^*}(y)dy \le \int_{|y|>T^\alpha}\left|\rho_{Y_n}\left(x-\frac{y}{T}\right)\right|dy = \left(1-\int_{|y|\le T^\alpha}\left|\rho_{Y_n}\left(x-\frac{y}{T}\right)\right|dy\right) \overset{u=x-\frac{y}{T}}{=}$$

$$= \left(1+T\int_{x+T^{\alpha-1}}^{x-T^{\alpha-1}}\left|\rho_{Y_n}(u)\right|du\right) = 1+T\left[F(x-T^{\alpha-1})-F(x+T^{\alpha+1})\right] = 1-T, \quad \text{if } T\to\infty.$$

Similarly, it is shown that $J = O(1-T)$.

□

The following theorem that establishes the rate of convergence of the probability density of the sequence of random variables (Y_n), towards the probability density of the $\chi^2(n,\sigma)$ distribution of one variation and $n=3$ degrees of freedom. Along the this section, we consider (Y_n^*) random variable which has the distribution $\chi^2(3,1)$.

Theorem 2.1. Let $(\rho_{Y_n}(x))$ be probability density Hölder continuous of the exponent $\gamma\in(0,1]$ for n sufficiently great. More than that, the following condition is met: there is a non-negative function $g(t)\in L^1(R)$ and a sequence (Δ_n), so that for $|t|\le\Delta_n$,

$$\left|\varphi_{Y_n}(t)-\frac{1}{\sqrt{(1-2it)^3}}\right| \le \frac{g(t)}{\Delta_n} \tag{1}$$

then exists $C>0$ such that

$$\sup_x\left|\rho_{Y_n}(x)-f_{Y_n^*}(x)\right| = C\left(\frac{2}{T}\right)^\gamma\cdot\frac{\Gamma\left(\gamma+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} + O\left[M + \left(1-\frac{\Delta_n}{K}\right)A(\Delta_n,T)+B(\Delta_n)\right]$$

if $n\to\infty$, where $(f_{Y_n^*}(x))$ is the probability density of the random variable (Y_n^*) which has the distribution $\chi^2(3,1)$ and $T>0$.

The following result is useful:

Theorem 2.2. We suppose that the characteristic function $\varphi_{Y_n}(x)$ meets the condition (1). There are the positive constants c_1 and c_2 such that for $\alpha>1$ and $T>0$, to have the inequality:

$$\left|\rho_{Y_n}(x)-f_{Y_n^*}(x)\right| = \int_{|y|\le T^\alpha}\left|\rho_{Y_n}\left(x-\frac{y}{T}\right)-\rho_{Y_n}(x)\right| f_{Y_n^*}(y)dy + \frac{1}{2\pi}\int_{|t|>\Delta_n}\left|\varphi_{Y_n}(t)\right|\frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}}dt + O\left(A(\Delta_n,T)\right)+O\left(B(\Delta_n)\right)+O(M_1).$$

Proof. According to $\rho_{Y_n}(x,T)$ definition,

$$\rho_{Y_n}(x,T)=\int_{-\infty}^{+\infty}\rho_{Y_n}\left(x-\frac{y}{T}\right)f_{Y_n^*}(y)dy,\ T>0.$$

From Lemma 2.1, results that

$$\left|\rho_{Y_n}(x,T)-\rho_{Y_n}(x)\right|=\int_{|y|\le T^{\alpha}}\left|\rho_{Y_n}\left(x-\frac{y}{T}\right)-\rho_{Y_n}(x)\right|f_{Y_n^*}(y)dy+O(1-T).$$

We note that $\rho_{Y_n}(x,T)$ is the probability density of random variable Y_n+N, where (Y_n) and N are independent random variables and $N\sim\chi^2\left(3,\frac{1}{T}\right)\cdot$ We have,

$$\left|\rho_{Y_n}(x)-f_{Y_n^*}(x)\right|\le\left|\rho_{Y_n}(x)-\rho_{Y_n}(x,T)\right|+\left|\rho_{Y_n}(x,T)-f_{Y_n^*}(x)\right|$$

We apply the reverse formula for $\left|\rho_{Y_n}(x,T)-f_{Y_n^*}(x)\right|$. By relation $\varphi_{Y_n+N}(t)=\int_{-\infty}^{+\infty}e^{itx}\rho_{Y_n}(x,T)dx$, we have that $\rho_{Y_n+N}(x,T)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-itx}\varphi_{Y_n}(t)\underbrace{\frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}}}_{\varphi_N(t)}dt.$

On the other hand, $\varphi_{Y_n^*}(t)=\int_{-\infty}^{+\infty}e^{itx}f_{Y_n^*}(x)dx$, from where $f_{Y_n^*}(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-itx}\frac{1}{\sqrt{(1-2it)^3}}dt.$ Then

$$\left|\rho_{Y_n}(x,T)-f_{Y_n^*}(x)\right|=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-itx}\left(\varphi_{Y_n}(t)\frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}}-\frac{1}{\sqrt{(1-2it)^3}}\right)dt.$$

and

$$\sup_x\left|\rho_{Y_n}(x,T)-f_{Y_n^*}(x)\right|\le\frac{1}{2\pi}\left(I_1+I_2+I_3+I_4\right)$$

where

$$I_1=\int_{|t|\le\Delta_n}\left|\varphi_{Y_n}(t)-\frac{1}{\sqrt{(1-2it)^3}}\right|\frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}}dt$$

$$I_2=\int_{|t|\le\Delta_n}\frac{1}{\sqrt{(1-2it)^3}}\left|1-\frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}}\right|dt$$

$$I_3 = \int_{|t|>\Delta_n} \left|\varphi_{Y_n}(t)\right| \frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}} dt$$

$$I_4 = \int_{|t|>\Delta_n} \frac{1}{\sqrt{(1-2it)^3}} dt$$

For the first integral we have the following statement if we take into account the function development $h(t) = \frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}}$ in series of power:

$$I_1 \overset{(1)}{\leq} \int_{|t|\leq\Delta_n} \frac{g(t)}{\Delta_n} \frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}} dt \leq \frac{K}{\Delta_n} \int_{|t|\leq\Delta_n} \frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}} dt \approx \frac{K}{\Delta_n} \int_{|t|\leq\Delta_n} \left(1+\frac{it}{T^2}-\frac{3t^2}{2T^4}\right)^3 dt =$$

$$= K\left[2-\frac{27}{28T^{12}}\Delta_n^6-\left(\frac{27}{10T^8}+\frac{3}{5T^6}\right)\Delta_n^4-\frac{5}{T^4}\Delta_n^2\right] \overset{not.}{=} A(\Delta_n, T),$$

or $I_1 = O\left(A(\Delta_n, T)\right)$.

For the second integral,

$$I_2 \leq \int_{|t|\leq\Delta_n} \frac{1}{\sqrt{(1-2it)^3}} dt \approx \int_{|t|\leq\Delta_n} \left(1+it-\frac{3}{2}t^2\right) dt = 2\Delta_n - \frac{27}{28}\Delta_n^7 - 5\Delta_n^3 + \frac{9}{2}\Delta_n^5 \overset{not.}{=} B(\Delta_n).$$

It results that $I_2 = O\left(B(\Delta_n)\right)$.

I_4 integral is an improper integral. We study its convergence by the help of a *convergence limit criterion*. As the integrant is positive in the domain $(-\infty, \Delta_n) \cup (\Delta_n, +\infty)$, there is

$$\lim_{t\to\infty} \frac{t^\alpha}{(1-2it)^{3/2}} = \begin{cases} \infty & \text{if} \quad \alpha > \frac{3}{2} \\ -\frac{1}{2i} & \text{if} \quad \alpha = \frac{3}{2} \\ 0 & \text{if} \quad \alpha < \frac{3}{2} \end{cases}.$$

The limit is different of zero and infinite when $\alpha = \frac{3}{2} > 1$. According to the convergence limit criterion of improper integrals on unlimited interval, it results that the integral is convergent. Let $I_4 = O(M_1)$ be it.

□

The proof of Theorem 2.1

According to Theorem 2.2 and to the Hölder continuous condition for the density probability $\rho_{Y_n}(x)$,

$$\int_{|y|\leq T^\alpha} \left|\rho_{Y_n}\left(x-\frac{y}{T}\right)-\rho_{Y_n}(x)\right| f_{Y_n^*}(y)dy \leq \frac{C}{T^\gamma} \int_{-\infty}^{\infty} |y|^\gamma f_{Y_n^*}(y)dy$$

where $f_{Y_n^*}$ is the density of probability of the random variable $\left(Y_n^*\right)$, with the distribution $\chi^2(3,1)$:

$$f_{Y_n^*}(y)=\begin{cases}0 & \text{if } y<0\\ \dfrac{1}{2\sqrt{2}\,\Gamma\left(\frac{3}{2}\right)}\sqrt{y}e^{-y/2} & \text{if } y\geq 0\end{cases}$$

$$\int\limits_{|y|\leq T^{\alpha}}\left|\rho_{Y_n}\left(x-\frac{y}{T}\right)-\rho_{Y_n}(x)\right|f_{Y_n^*}(y)dy\leq\frac{C}{2\sqrt{2}\,\Gamma\left(\frac{3}{2}\right)T^{\gamma}}\int_0^{\infty}|y|^{\gamma+1/2}e^{-y/2}dy$$

$$\overset{y/2=t}{=}\frac{C2^{\gamma+3/2}}{2\sqrt{2}\,\Gamma\left(\frac{3}{2}\right)T^{\gamma}}\int_0^{\infty}t^{\gamma+1/2}e^{-t}dt=\frac{C2^{\gamma+3/2}}{2\sqrt{2}\,\Gamma\left(\frac{3}{2}\right)T^{\gamma}}\Gamma\left(\gamma+\frac{3}{2}\right)=C\left(\frac{2}{T}\right)^{\gamma}\frac{\Gamma\left(\gamma+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

Regarding the third integral,

$$I_3=\int\limits_{|t|>\Delta_n}\left|\varphi_{Y_n}(t)\right|\frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}}dt\leq\int\limits_{|t|>\Delta_n}\frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}}dt$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}}dt-\int\limits_{|t|\leq\Delta_n}\frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}}dt$$

$$\leq O(M_2)-C(\Delta_n,T),$$

where, $C(\Delta_n,T)=\left[2\Delta_n-\frac{27}{28T^{12}}\Delta_n^7-\left(\frac{27}{10T^8}+\frac{9}{5T^6}\right)\Delta_n^5-\frac{5}{T^4}\Delta_n^3\right]$

and $I^*=\int_{-\infty}^{\infty}\frac{1}{\sqrt{\left(1-\frac{2it}{T^2}\right)^3}}dt$ is a convergent integral ($I^*=O(M_2)$). It is easily observed that

$C(\Delta_n,T)=A\left(\Delta_n,T\right)\cdot\frac{\Delta_n}{K}$.

So,

$$\left|\rho_{Y_n}(x,T)-f_{Y_n^*}(x)\right|=C\left(\frac{2}{T}\right)^{\gamma}\frac{\Gamma\left(\gamma+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}+O\left(M+\left(1-\frac{\Delta_n}{K}\right)A(\Delta_n,T)+B\left(\Delta_n\right)\right),$$

where $M=M_1+M_2$.

□

Such a study, that remains open, could be made starting from the Theorem of Esseen, in his paper [1]:

Theorem 2.3. ([1]) Let $F(x)$ and $G(x)$ be two distribution functions with adequate characteristic functions f and g. We suppose that $G(x)$ has the derivative of first order, bounded so that . Then, for any $T > 0$ and forall $b > 1/(2\pi)$ we have

$$\sup_x |F(x) - G(x)| \leq b \int_{-T}^{T} \left| \frac{f(t) - g(t)}{t} \right| dt + c(b) \frac{K}{T},$$

where $c(b)$ is a positive constant that depends on b.

This theorem has a fundamental role in estimating the rate of convergence in the central limit theorem. These results have been shown in the author's paper [3].

References:

1. Esseen C.G. On the remainder term in the central limit theorem // Arkiv. Mat., 1968, vol.8, no.1, p.7-15.
2. Macht W. and WOLF, W. On the local central limit theorem // Limit theorems in probability theory and related fields, 1987, p.69-87.
3. Munteanu B.GH. On a Generalization of Esseen`s inequality // Italian Journal of Pure and Applied Mathematics, 2008, vol.24, p.83-90.

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