METHOD OF "CATASTROPHES" AND ITS APPLICATION TO ANALYZE GENERALIZED QUEUEING MODELS

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Metoda „catastrofelor" se bazează pe tehnica transformatelor Laplace şi Laplace-Stieltjes. Această metodă analizează evoluţia sistemului în baza unor evenimente aleatorii, numite „catastrofe", datorită cărui fapt se reuşeşte de a atribui un anumit sens probabilist transformatelor menţionate [1].

În această lucrare, metoda „catastrofelor" se aplică pentru a obţine caracteristicile probabilistice de evoluţie a modelelor de aşteptare generalizate. Vor fi prezentate demonstraţii pentru repartiţia timpului total de servire a cererilor de prioritate *k* şi a timpului total de trecere la clasa de prioritate *k* pentru diferite scheme prioritare. Aceste caracteristici îşi găsesc aplicare în analiza evoluţiei reţelelor contemporane de bandă largă fără fir [2,3], în analiza şi optimizarea Centrelor de Apel [3,4] etc. Repartiţiile sunt obţinute în termenii transformatei Laplace-Stieltjes. Demonstraţiile sunt ilustrate prin desene.

1. Introduction

Analytical methods in queueing theory are based on Laplace and Laplace-Stieltjes transform technique, *Z*-transform, Markovian processes, stochastic calculus, martingales theory, matrix transformations, etc.

In this paper we will present some analytical results obtained with Laplace and Laplace-Stieltjes transform technique.

1.1. Laplace and Laplace-Stieltjes transforms

In this subsection we will present preliminary results about Laplace and Laplace-Stieltjes transforms, because are techniques used in method of "catastrophes". Laplace and Laplace-Stieltjes transforms are characterized by remarkable properties that find their application in optics, operational calculus, functional analysis, probability theory, etc.

Let $A(t)$ be a complex-valued function of real argument satisfying the following conditions:

- $A(t)=0,\ \forall t<0$, and $A(t)$ is a function with bounded variation on any segment $[0, T]$;
- $\exists s_0,\ M \in R, s.t. |A(t)| \le Me^{-s_0 t}$.

Definition 1.1. The Laplace transform of a function $A(t)$, for $Re(s) > s_0$, is given by integral:

$$\overline{\alpha}(s) = \int_0^\infty e^{-st} A(t)dt$$

It is denoted by $\overline{\alpha}(s)$ or $\Lambda[A(t)](s)$.

The Laplace transform has the following properties:

Proposition 1.1. The Laplace transform is analytical in half-plane $Re(s) > s_0$.

Proposition 1.2. The Laplace transform is unique. Let be $A_1(t)$ and $A_2(t)$ two functions with Laplace transform $\overline{\alpha}_1(s)$, and respectively $\overline{\alpha}_2(s)$, and $\overline{\alpha}_1(s)=\overline{\alpha}_2(s)$ for $Re(s) > s_0$, then $A_1(t)=A_2(t)$ in all continuity points of function $A_1(t)$ and $A_2(t)$.

Proposition 1.3. The Laplace transform is linear. Let be $\overline{\alpha}(s)$ – Laplace transform of function $A(t)$ and $\overline{\beta}(s)$ – Laplace transform of function $B(t)$. If $C(t) = aA(t)+bB(t)$ then $\overline{c}(s) = a\overline{\alpha}(s) + b\overline{\beta}(s)$.

Proposition 1.4. The Laplace transform of a convolution. If $C(t) = A(t)\cdot B(t) = \int_0^\infty A(t-\tau)B(\tau)d\tau$ then $\overline{c}(s) = \overline{\alpha}(s)\cdot\overline{\beta}(s)$.

Let be *A* a random positive variable and *A(t)* its distribution function (a real-valued function with real argument).

Definition 1.2. The Laplace-Stieltjes transform of function *A(t)* call the Lebesgue-Stieltjes integral:

$$\alpha(s) = \int_0^\infty e^{-st} dA(t),$$

where $s \in C$.

It is denoted by $\alpha(s)$ or $\Lambda S[A(t)](s)$.

The following identity has a big importance, because make the connection between Laplace-Stieltjes and Laplace transform:

If exist $\lim_{t\downarrow 0} A(t)$, then $\lim_{t\downarrow 0} A(t) = \lim_{s\to+\infty} \alpha(s)$

$$\alpha(s) = s\int_0^\infty e^{-st} A(t)dt = s\overline{\alpha}(s)\cdot$$

Let *X* is a random variable with distribution function $F_X(t)$, s.t.

$$F_X(t) = P(X \le t),$$

then Laplace-Stieltjes transform of a random variable *X* can be represented as:

$$\Lambda S[f_X(t)] = E[e^{-sX}].$$

1.2. The "catastrophes" method. Probabilistic interpretation of Laplace-Stieltjes transform

The methods of "catastrophes" in the simplest version can be found in works of Danzig, he introduced the probabilistic interpretation of Laplace Stieltjes transform. In the Queueing Theory this method was first used by Klimov G.P. to study classical queueing systems, by Gnedenco B.V, Danielean A.A, Dimitrov B. and Ivanov G.A. to study priority queueing systems and by Klimov G.P and Mishkoy Gh. to study queueing system with orientation [5-8].

The "catastrophes" method consists in introducing of supplementary random event. In this sense we can associate a well defined probabilistic sense to mathematical structures such as Laplace transform, generating function, etc. which then can be used to determine the probability of more complicated events.

The essence of the method of introducing supplementary random event can be presented by the following lemma.

Lemma 1.1. Laplace-Stieltjes transform value of distribution function of the positive random variable *A*, for *s > 0* is equal to the probability that during the realization of random variable *A*, there has been no message of Poisson flow with parameter *s*, i.e.

$$Me^{-sA} = P(A < x)$$

For the demonstration we should mention that

$$\int_0^\infty e^{-st} dA(t) = P\{A < x\}$$

and $dA(t) = P(A \in [t, t+dt])$.

The following theorem is a cornerstone of the method of "catastrophes" [9].

Theorem 1.1. Let *X* and *Y* be two independent random variable. Suppose that *Y* is exponential distributed, i.e. *Y ~ Exp(s)*, and the density function of variable *X* is $f_X(t)$. Then

$$P(X < Y) = \Lambda[f_X(t)](s) = \Lambda S[X].$$

We present an example of direct application of "catastrophes" method, for queueing system of type M |G|1.

Example 1.1. (Kendall equation) Consider the system M |G|1 Poisson arrival flows, with rate λ and random service time *B* with distribution function *B(t)*. First, note that busy period and arrival moments are independent random variable identically distributed, with some distribution function, otherwise we can't apply Laplace transform. Let *β(s)* be Laplace-Stieltjes transform of function *B(t)* and *π(s)* Laplace-Stieltjes transform of function *Π(t)*. Using the "catastrophes" method we can show that:

$$\pi(s) = \beta(s + \lambda(1 - \pi(s))).$$

Proof.

$$\pi(s)=\sum_{k\geq 0}[\pi(s)]^k\int_0^\infty e^{-st}\frac{(\lambda t)^k}{k!}e^{-\lambda t}dB(t)=\sum_{k\geq 0}\int_0^\infty e^{-(s+\lambda)t}\frac{(\lambda\pi(s)t)^k}{k!}dB(t)=$$

$$=\int_0^\infty e^{-(s+\lambda)t}e^{\lambda\pi(s)t}dB(t)=\int_0^\infty e^{-(s+\lambda-\lambda\pi(s))t}dB(t)=\beta(s^{'})$$

where $s^{'}=s+\lambda-\lambda\pi(s)$.

2. Some probabilistic characteristics of priority queueing system with switchover time

2.1. Description of system

Let be a queueing system with r independent poissonian flows $L_1, \ldots, L_r$, with parameters $a_1, \ldots, a_r$ respectively. Period of service requests of flow L_k is a random variable B_k with distribution function $B_k(x)$, $k = 1, \ldots, r$. The system simultaneously can serve no more than one request, and if the system has served a request of a flow L_i, that it could begin service of a request of a flow L_k, $i = k$, it is required to spend some time C_k for orientation of the system. Duration of orientation from L_i to L_k ($i = 1, \ldots, r;\ i \neq k$) is a random variable with distribution function $C_k(x)$.

Preemptive priority. It means, that if in the system arrive the request of the highest priority from available in system then it interrupt orientation to service and service of a request and the system begin orientation to service the higher priority request. The following disciplines of orientation and service for preemptive priority can be considered.

The scheme 1.1.

a) if during orientation of the system from L_i to L_k $(\rightarrow k)$ the request of a flow L_j arrives, $j<k$ then orientation $(\rightarrow k)$ is interrupted and at once orientation $(\rightarrow j)$ begins. When the system will be free from requests of a priority above k, the interrupted orientation $(\rightarrow k)$ begins anew (with new realization of time of orientation);

b) if during service of a request of a flow L_k the request of a flow L_j arrives, $j<k$ then the service is interrupted, orientation at once begins $(\rightarrow j)$ and as soon as it is finished, service of the request led interruption begins. As soon as the system will be released (free) from requests of a priority above k, orientation $(\rightarrow k)$ begins. When orientation $(\rightarrow k)$ is finished, service of the interrupted request begins anew (with new realization of a service time).

The scheme 1.2.

a) the same, as a) schemes 1.1;

b) the same, as b) schemes 1.1, but the request with the interrupted service "is lost".

The scheme 1.3.

a) the same, as a) schemes 1.1;

b) the same, as b) schemes 1.1, but the request with the interrupted service is served remained time.

The scheme 1.4.

a) the same, as a) schemes 1.1;

b) the same, as b) schemes 1.1, but the service of the interrupted request begins anew with former realization of a service time (identical service anew).

The scheme 2.1.

a) the same, as a) schemes 1.1, but the interrupted orientation is oriented remained time;

b) the same, as b) schemes 1.1.

The scheme 2.2.

a) the same, as a) schemes 2.1;

b) the same, as b) schemes 1.2.

The scheme 2.3.

a) the same, as a) schemes 2.1;

b) the same, as b) schemes 1.3.

The scheme 2.4.
a) the same, as a) schemes 2.1;
b) the same, as b) schemes 1.4.
The scheme 3.1.
a) the same, as a) schemes 1.1, but the interrupted orientation $(\to k)$ begins anew with the same realization of time of orientation (identical orientation anew);
b) the same, as b) schemes 1.1.
The scheme 3.2.
a) the same, as a) schemes 3.1;
b) the same, as b) schemes 1.2.
The scheme 3.3.
a) the same, as a) schemes 3.1;
b) the same, as b) schemes 1.3.
The scheme 3.4.
a) the same, as a) schemes 3.1;
b) the same, as b) schemes 1.4.

Definition 2.1. k - cycle of orientation – begins from the moment of the beginning of orientation of the device to service a_k - request; comes to an end at once as soon as the device is ready to start service of this request. Duration k - cycle of orientation is a random variable N_k with distribution function $N_k(x)$ Laplace-Stieltjes transform, which is $\nu_k(s)$.

Definition 2.2. k - service cycle – begins from the moment of the beginning of service a_k - request; comes to an end at once as soon as the system will be released from this request. Duration k - service cycle is a random variable H_k with distribution function. $H_k(x)$ Laplace-Stieltjes transform which is $h_k(s)$.

Definition 2.3. Π_k - period – begins from the moment of receipt a_i - request $i < k$ in free system; comes to an end at once as soon as the system is released from a_k - requests. Duration Π_k - period is a random variable Π_k with distribution function $\Pi_k(x)$, Laplace-Stieltjes transform which is $\pi_k(s)$.

In this section for specified above schemes, are obtained distributions of k - cycles of orientation and k - service cycles.

The structure of k - cycle of orientation and k - cycle of service are illustrated with drawings. The common designs notations are specified in Fig.1. In demonstrations the expression "probability of this event is ..." is noted by abbreviation < ... >.

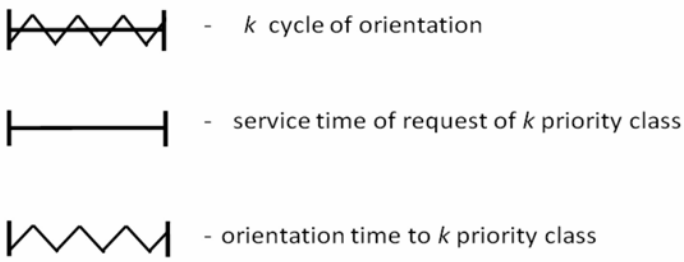

Fig.1. Significance of the designs notations

Let be $\nu_k(s)$ Laplace-Stieltjes transform of distribution functions of length k - cycles of orientation and $h_k(s)$ Laplace-Stieltjes transform of distribution function of length k - service cycles. For the described system take place the following expressions.

Lemma 2.1. For schemes 1.1, 2.1 and 3.1

$$h_k(s) = \beta_k(s+\sigma_{k-1})\{1 - \frac{\sigma_{k-1}}{s+\sigma_{k-1}}[1-\beta_k(s+\sigma_{k-1})]\pi_{k-1}(s)\nu_k(s)\}^{-1}$$

Proof. Consider that independent of the system evolution occurring some events, named "catastrophes", which forms a Poisson flow with parameter $s>0$. Consider also that during k– service cycle have not occurred "catastrophes". Probability of this event is – $h_k(s)$.

For it is necessary and sufficient that:

– or there was finished service of *k* priority request and during its have not occurred "catastrophes" – $<\beta_k(s+\sigma_{k-1})>$

– or during unfinished service of such request have not occurred "catastropphes" - $<\frac{\sigma_{k-1}}{s+\sigma_{k-1}}[1-\beta_k(s+\sigma_{k-1})]>$, have not occurred "catastrophes" during Π_{k-1} period - $<\pi_{k-1}(s)>$, and also have not occurred "catastrophes" during the *k*– cycle of orientation - $<\nu_k(s)>$.

The structure of *k*– cycle of service for mentioned schemes 1.1, 2.1 and 3.2 is presented on Fig.2.

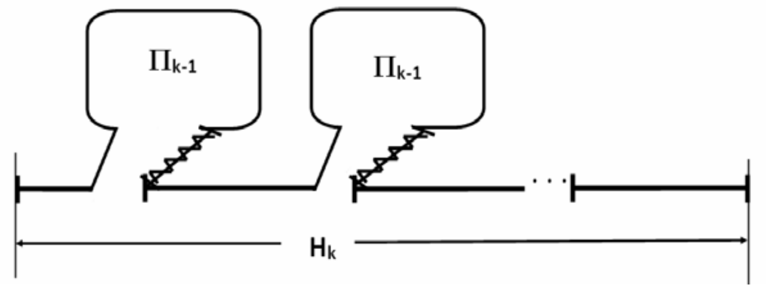

Fig.2

The demonstration of the following Lemmas 2.2 - 2.7 are analogically. The sketch of the proof of the mentioned Lemmas is the following. We consider that during *k*– cycle of service (*k*– cycle of orientation) have not occurred "catastrophes". The probability of this event is $h_k(s)$ $(\nu_k(s))$. Then the same probabilities are determined using the structure of concrete scheme.

Lemma 2.2. For schemes 1.2, 2.2 and 3.2

$$h_k(s)=\beta_k(s+\sigma_{k-1})+\frac{\sigma_{k-1}}{s+\sigma_{k-1}}[1-\beta_k(s+\sigma_{k-1})]\pi_{k-1}(s)\nu_k(s)$$

The structure of *k* – cycle of service for schemes 1.1, 2.1 and 3.2 is presented on Fig.3.

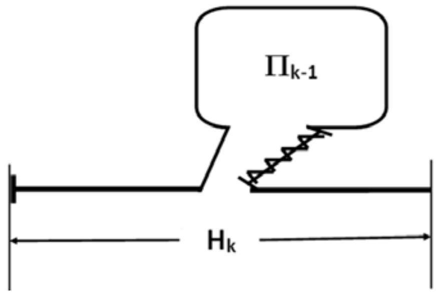

Fig.3

Lemma 2.3. For schemes 1.3, 2.3 and 3.3

$$h_k(s)=\beta_k(s+\sigma_{k-1})[1-\pi_{k-1}(s)\nu_k(s)]$$

The structure of *k* – cycle of service for schemes 1.3, 2.3 and 3.3 is presented on Fig.4.

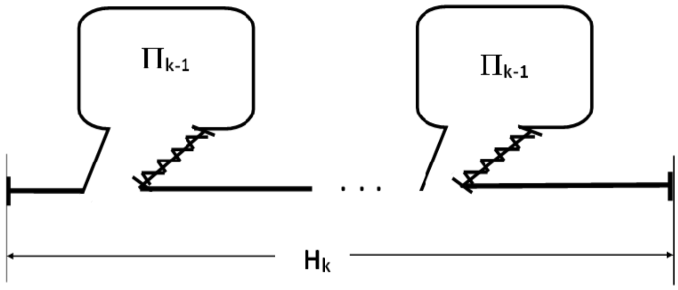

Fig.4

We note by D the period of time, beginning with $(k \to i)$, $i < k$ orientation and ending when the system is ready to continue the serving of interrupted request.

Lemma 2.4. For schemes 1.4, 2.4 and 3.4

$$h_k(s) = (s + \sigma_{k-1}) \int_0^\infty \frac{e^{-(s+\sigma_{k-1})u} dB_k(u)}{s + \sigma_{k-1}\{1 - \pi_{k-1}(s)\nu_k(s)[1 - e^{-(s+\sigma_{k-1})u}]\}}$$

Lemma 2.5. For schemes 1.1-1.4

$$\nu_k(s) = c_k(s + \sigma_{k-1})\left\{1 - \frac{\sigma_{k-1}}{s + \sigma_{k-1}}\left[1 - c_k(s + \sigma_{k-1})\right]\pi_{k-1}(s)\right\}$$

The structure of k – cycle of orientation for schemes 1.1 - 1.4 is presented on Fig.5.

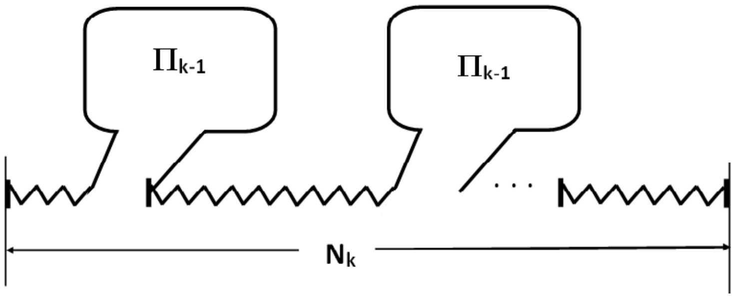

Fig.5

We note by M the period of time, beginning with $(i \to k)$, $i < k$ orientation and ending when the system becomes free of requests of type $a_1, \ldots, a_{k-1}$, in the assumption that during $(i \to k)$ orientation arrive requests of type $a_1, \ldots, a_{k-1}$.

Lemma 2.6. For schemes 2.1-2.4

$$\nu_k(s) = c_k(s + \sigma_{k-1}[1 - \pi_{k-1}(s)]) .$$

The structure of k – cycle of orientation for schemes 2.1 - 2.4 is presented on Fig.6.

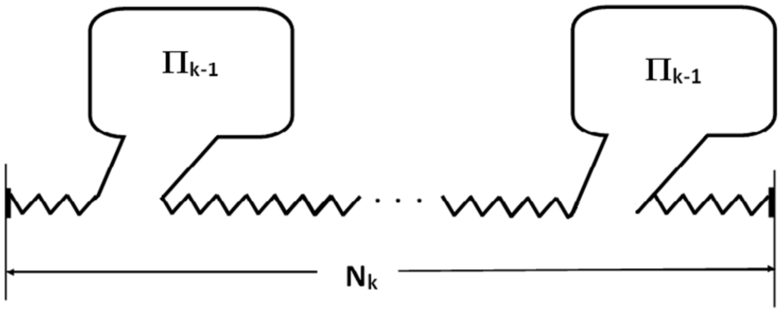

Fig.6

Lemma 2.7. For schemes 3.1-3.4.

$$\nu_k(s) = (s + \sigma_{k-1}) \int_0^\infty \frac{e^{-(s+\sigma_{k-1})r} dC_k(r)}{s + \sigma_{k-1}\{1 - \pi_{k-1}(s)[1 - e^{-(s+\sigma_{k-1})r}]\}}$$

Conclusion

In this paper, method of "catastrophes" was applied to obtain some probabilistic characteristics of evolution priority queuing models with switchover time. Namely the distributions of length of k–cycle of service and k–cycle of orientation, for different schemes of service and orientation were obtained. The distributions were obtained in terms of Laplace-Stieltjes transform. We can use this method also for obtain the others probabilistic characteristics of priority queueing systems, such as busy period of system with k priority requests and higher than *k*, etc. These characteristics find their application in various practical problems, for example in analyzing the evolution of contemporary broadband wireless networks [2,3], analysis and optimization of Call centers [3,4].

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