SOME PROPERTIES ON TOPOLOGICAL PARAMEDIAL GROUPOIDS WITH MULTIPLE IDENTITIES

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În acest articol sunt studiate unele proprietăți ale (n,m)-izotopiilor omogene ale grupoizilor topologici paramediali cu (n,m)-unități. Au fost extinse unele afirmații ale teoriei grupurilor topologice în clasa quasigrupurilor topologice paramediale.

1. Introduction

In this article we study the (n,m)-homogeneous isotopies of paramedial topological groupoid with multiple identities and measure Haar on paramedial topological quasigroups. The results established in this paper are related to the results of M. Choban and L. Kiriyak in [1] and to the research papers [2-7]. In section 4 we expand on the notions of multiple identities and (n,m)-homogeneous isotopies introduced in [2]. This concept facilitates the study of topological groupoids with (n,m)-identities. In this section we prove that if $(G,+)$ is a paramedial topological groupoid and e is a (k,p)-zero, then every (n,m)-homogeneous isotope $(G,\cdot)$ of $(G,+)$ is a paramedial topolgical groupoid, with (mk,np) - identity e in $(G,\cdot)$. In section 5 we study the direct products of groupoids with multiple identities. In this context we prove some assertions on the direct products. In section 6 we mention some remarks on Haar measures on paramedial topological quasigroups. We shall use the notations and terminology from [1-3, 13].

2. Basic notions

A non-empty set G is said to be a *groupoid relatively* to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a,b) of elements of G there is a unique element $ab \in G$.

If the groupoid G is a topological space and the binary operation $(a,b) \to a \cdot b$ is continuous, then G is called a *topological groupoid*.

An element $e \in G$ is called an *identity* if $ex = xe = x$ for every $x \in X$.

A quasigroup with an identity is called a *loop*.

A groupoid G is called *medial* if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x,y,z,t \in G$.

A groupoid G is called *paramedial* if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x,y,z,t \in G$.

A groupoid G is called *bicommutative* if it satisfies the law $xy \cdot zt = tz \cdot yx$ for all $x,y,z,t \in G$.

If a paramedial quasigroup G contains an element e such that $e \cdot x = x$ $(x \cdot e = x)$ for all x in G, then e is called a *left (right) identity* element of G and G is called a *left (right) paramedial loop.*

3. Groupoids with multiple identities

Consider a groupoid $(G,+)$. For every two elements a,b from $(G,+)$ we denote:

$1(a,b,+) = (a,b,+)1 = a + b$ and $n(a,b,+) = a + (n-1)(a,b,+)$, $(a,b,+)n = (a,b,+)(n-1) + b$ for all $n \geq 2$.

If a binary operation $(+)$ is given on a set G, then we shall use the symbols $n(a,b)$ and $(a,b)n$ instead of $n(a,b,+)$ and $(a,b,+)n$.

Definition 3.1 *Let $(G,+)$ be a groupoid and let $n,m \geq 1$. The element e of the groupoid $(G,+)$ is called:*

- an (n,m)-zero of G if $e + e = e$ and $n(e,x) = (x,e)m = x$ for every $x \in G$;

- *an* (n,∞)*-zero if* $e+e=e$ *and* $n(e,x)=x$ *for every* $x\in G$;
- *an* (∞,m)*-zero if* $e+e=e$ *and* $(x,e)m=x$ *for every* $x\in G$.

Clearly, if $e\in G$ is both an (n,∞)-zero and an (∞,m)-zero, then it is also an (n,m)-zero. If $(G,\cdot)$ is a multiplicative groupoid, then the element e is called an (n,m)-identity. The notion of (n,m)-identity was introduced in [2].

Example 3.1 *Let* $(G,\cdot)$ *be a paramedial groupoid,* $e\in G$ *and* $xe=x$ *for every* $x\in G$. *Then* $(G,\cdot)$ *is paramedial groupoid with* $(2,1)$*-identity* e *in* G. *Indeed, if* $x\in G$, *then* $e\cdot ex=ee\cdot ex=xe\cdot ee=xe\cdot e=xe=x$.

Example 3.2 Let $G=\{1,2,3,4,5,6,7,8\}$. We define the binary operation $\{\cdot\}$.

$(\cdot)$	1	2	3	4	5	6	7	8
1	1	2	3	4	7	8	5	6
2	2	1	4	3	8	7	6	5
3	5	6	7	8	3	4	1	2
4	6	5	8	7	4	3	2	1
5	3	4	1	2	5	6	7	8
6	4	3	2	1	6	5	8	7
7	7	8	5	6	1	2	3	4
8	8	7	6	5	2	1	4	3

Then $(G,\cdot)$ is a paramedial and non-medial quasigroup. For example, $(4\cdot 3)\cdot(2\cdot 5)\neq(4\cdot 2)\cdot(3\cdot 5)$. In this case we have that 1 and 5 are $(2,2)$- identities in $(G,\cdot)$.

4. Homogeneous isotopies

Definition 4.1 *Let* $(G,+)$ *be a toplogical groupoid. A groupoid* $(G,\cdot)$ *is called a homogeneous isotope of the topological groupoid* $(G,+)$ *if there exist two topological automorphisms* $\varphi,\psi:(G,+)\to(G,+)$ *such that* $x\cdot y=\varphi(x)+\psi(y)$, *for all* $x,y\in G$.

For every mapping $f:X\to X$ *we put* $f^1(x)=f(x)$ *and* $f^{n+1}(x)=f(f^n(x))$ *for any* $n\geq 1$.

Definition 4.2 *Let* $n,m\leq\infty$. *A groupoid* $(G,\cdot)$ *is called an* (n,m)*-homogeneous isotope of a topological groupoid* $(G,+)$ *if there exist two topological automorphisms* $\varphi,\psi:(G,+)\to(G,+)$ *such that:*

1. $x\cdot y=\varphi(x)+\psi(y)$ *for all* $x,y\in G$;
2. $\varphi\varphi=\psi\psi$;
3. *If* $n<\infty$, *then* $\varphi^n(x)=x$ *for all* $x\in G$;
4. *If* $m<\infty$, *then* $\psi^m(x)=x$ *for all* $x\in G$.

Definition 4.3 *A groupoid* $(G,\cdot)$ *is called an isotope of a topological groupoid* $(G,+)$, *if there exist two homeomorphisms* $\varphi,\psi:(G,+)\to(G,+)$ *such that* $x\cdot y=\varphi(x)+\psi(y)$ *for all* $x,y\in G$.

Under the conditions of Definition 4.3 we shall say that the isotope $(G,\cdot)$ is generated by the homeomorphisms φ,ψ of the topological groupoids $(G,+)$ and denote $(G,\cdot)=g(G,+,\varphi,\psi)$.

Theorem 4.1 *If* $(G,+)$ *is a paramedial topological groupoid and* $e\in G$ *is a* (k,p)*-zero, then every* (n,m)*-homogeneous isotope* $(G,\cdot)$ *of the topological groupoid* $(G,+)$ *is a paramedial groupoid with* (mk,np)*-identity* e *in* $(G,\cdot)$ *for all* $m,k,n,p\in N$.

Proof. Let e be a (k,p)-zero in $(G,+)$ and $(G,\cdot)$ be an (n,m)-homogeneous isotope of the groupoid $(G,+)$. We will prove that e is an (mk,np)-identity in $(G,\cdot)$. We mention that $\varphi^r(e)=\psi^r(e)=e$ for every $r\in N$. If $k<+\infty$, then in $(G,+)$ we have $rk(e,x,+)=x$ for each $x\in G$ and for every $r\in N$. Let $m<+\infty$ and $\psi^m(x)=x$ for all $x\in G$. Then $1(e,x,\cdot)=1(e,\psi(x),+)$ and $r(e,x,\cdot)=r(e,\varphi^r(x),+)$ for every $r\geq 1$.

Therefore

$$mk(e,x,\cdot)=mk(e,\psi^{mk}(x),+)=mk(e,x,+)=x.$$

Analogously we obtain that

$$(e,x,\cdot)np=(e,\varphi^{np}(x),+)np=(e,x,+)np=x.$$

Hence, e is an (mk,np)-identity in $(G,\cdot)$.

We will prove that (n,m)-homogeneous isotope $(G,\cdot)$ of the paramedial topological groupoid $(G,+)$ is paramedial topological groupoid and $xy\cdot zt=ty\cdot zx$. Really, using well known algorithm, we obtain

$$\begin{aligned}xy\cdot zt&=\varphi(xy)+\psi(zt)=\varphi(\varphi(x)+\psi(y))+\psi(\varphi(z)+\psi(t))=\\&=[\varphi(\varphi(x))+\varphi(\psi(y))]+[\psi(\varphi(z))+\psi(\psi(t))]=[\psi(\psi(t))+\varphi(\psi(y))]+[\psi(\varphi(z))+\varphi(\varphi(x))]=\\&=[\varphi(\varphi(t))+\varphi(\psi(y))]+[\psi(\varphi(z))+\psi(\psi(x))]=\varphi(\varphi(t)+\psi(y))+\psi(\varphi(z)+\psi(x))=\\&=\varphi(t\cdot y)+\psi(z\cdot x)=ty\cdot zx.\end{aligned}$$

Hence, $(G,\cdot)$ is a paramedial topological groupoid with (mk,np)-identity e.

The proof is complete.

5. Direct products of groupoids with multiple identities

Definition 5.1 *The direct product $Q_1\times Q_2\times\ldots\times Q_n$ of the groupoids $Q_1,Q_2,\ldots,Q_n$ with operations $\circ_1,\circ_2,\ldots,\circ_n$, respectively, is the set of n - tuples $(q_1,q_2,\ldots,q_n)$ where $q_i\in Q_i$ with operation defined componentwise:*

$$(q_1,q_2,\ldots,q_n)*(h_1,h_2,\ldots,h_n)=(q_1\circ_1 h_1,q_2\circ_2 h_2,\ldots,q_n\circ_n h_n).$$

The least multiple common of a and b we denote by $c(a,b)$.

We will examine the direct product of the groupoids $(Q_1,\cdot)$ and $(Q_1,\circ)$ with multiple identities.

Theorem 5.1 *Let $(Q_1,\cdot)$ be a groupoid with an (n,m)-identity and $(Q_2,\circ)$ be a groupoid with a (k,l)-identity. Then their direct product $G=Q_1\times Q_2$ is a groupoid with a $(c(n,k),c(m,l))$-identity. Furthermore:*

1) if $(Q_1,\cdot)$ and $(Q_2,\circ)$ are medial, then G is medial too;

2) if $(Q_1,\cdot)$ and $(Q_2,\circ)$ are paramedial, then G is paramedial too;

3) if $(Q_1,\cdot)$ and $(Q_2,\circ)$ are bicommutative, then G is bicommutative too.

Proof. Let e_1 be an (n,m)-identity of $(Q_1,\cdot)$ and e_2 be a (k,l)-identity of groupoid $(Q_2,\circ)$. It is clear that in $(Q_1,\cdot)$ we have $e_1\cdot e_1=e_1$, $n(e_1,x)=x$ and $(x,e_1)m=x$. Similarly in the groupoid $(Q_2,\circ)$ we have $e_2\circ e_2=e_2$, $n(e_2,x)=x$ and $(x,e_2)l=x$. We will prove that $(e_1\cdot e_2)$ is $(c(n,k),c(m,l))$-identity in $G=Q_1\times Q_2$. Really,

1. $(e_1,e_2)(e_1,e_2)=(e_1\cdot e_1,e_2\circ e_2)=(e_1,e_2)$. Hence, (e_1,e_2) is identity;
2. $[(x,y),(e_1,e_2)]c(n,k)=[(x\cdot e_1,y\circ e_2)(e_1,e_2)](c(n,k)-1)=$

$$\begin{aligned}&=[((x,e_1)\cdot e_1,(y\cdot e_2)\circ e_2)(e_1,e_2)](c(n,k)-2)=\ldots=\\&=[((x\cdot e_1)\cdot(c(n,k)-1),(y\cdot e_2)\circ(c(n,k)-1)\ )](e_1,e_2)=\\&=[(x\cdot e_1)\cdot c(n,k),(y\cdot e_2)\circ c(n,k)]=(x,y);\end{aligned}$$

3. $c(m,l)[(e_1,e_2),(x,y)] = (c(m,l)-1)[(e_1,e_2)(e_1\cdot x, e_2\circ y)] =$
$= (c(m,l)-2)[(e_1,e_2)(e_1\cdot(e_1,x), e_2\circ(e_2,y))] = \ldots =$
$= (e_1,e_2)[((c(m,l)-1)\cdot(e_1,x)),(c(m,l)-1)\circ(e_2,y)] =$
$= [(c(m,l)\cdot(e_1,x)),(c(m,l)\circ(e_2,y))] = (x,y).$

Hence, $(e_1\cdot e_2)$ is $(c(n,k),c(m,l))$- identity in $Q_1\times Q_2$. The assertions 1-3 follows from well - known Birkhoff `s Theorem.
The proof is complete.

Corollary 5.1 *If* $(Q_1,\cdot)$ *and* $(Q_2,\circ)$ *are groupoids with an* $(1,m)$*-identity and an* $(n,1)$*-identity respectively, then their direct product* $G = Q_1\times Q_2$ *is a groupoid with an* (n,m)*-identity.*

Proof. Follows from Theorem **5.1**.

Theorem 5.2 *Let* $(Q_1,\cdot)$ *be a groupoid with* n *multiple identities* $(k_1,l_1),(k_2,l_2),\ldots,(k_n,l_n)$ *and* $(Q_2,\circ)$ *be a groupoid with* t *multiple identities* $(m_1,r_1),(m_2,r_2),\ldots,(m_t,r_t)$. *Then their direct product* $G = Q_1\times Q_2$ *has* $n\times t$ *multiple identities of following types:*

1) type of identities	2) type of identities	$\cdots$	n) type of identities
1. $(c(k_1,m_1),c(l_1,r_1))$	1. $(c(k_2,m_1),c(l_2,r_1))$	$\ldots$	1. $(c(k_n,m_1),c(l_n,r_1))$
2. $(c(k_1,m_2),c(l_1,r_2))$	2. $(c(k_2,m_2),c(l_2,r_2))$	$\ldots$	2. $(c(k_n,m_2),c(l_n,r_2))$
$\ldots\ldots\ldots\ldots$	$\ldots\ldots\ldots\ldots$		$\ldots\ldots\ldots\ldots$
t. $(c(k_1,m_t),c(l_1,r_t))$	t. $(c(k_2,m_t),c(l_2,r_t))$	$\ldots$	t. $(c(k_n,m_t),c(l_n,r_t))$

Proof. Let $(Q_1,\cdot)$ be a groupoid with n multiple identities $(k_1,l_1),(k_2,l_2),\ldots,(k_n,l_n)$ and $(Q_2,\circ)$ be a groupoid with t multiple identities $(m_1,r_1),(m_2,r_2),\ldots,(m_t,r_t)$. We examine the (k_1,l_1)-identity in $(Q_1,\cdot)$ and all multiple identities from $(Q_2,\circ)$. Using the Theorem 5.1 we obtain the first type of identities:

1. $(c(k_1,m_1),c(l_1,r_1))$, 2. $(c(k_1,m_2),c(l_1,r_2))$, $\ldots$, t. $(c(k_1,m_t),c(l_1,r_t))$.

For (k_2,l_2)-identity in $(Q_1,\cdot)$ and all multiple identities from $(Q_2,\circ)$ we apply Theorem 5.1 and we get the second type of identities:

1. $(c(k_2,m_1),c(l_2,r_1))$, 2. $(c(k_2,m_2),c(l_2,r_2))$, $\ldots$, t. $(c(k_2,m_t),c(l_2,r_t))$.

Continuing this process, in finally, for (k_n,l_n)-identity in $(Q_1,\cdot)$ and all multiple identities from $(Q_2,\circ)$, we obtain the $n-th$ type of identities:

1. $(c(k_n,m_1),c(l_n,r_1))$, 2. $(c(k_n,m_2),c(l_n,r_2))$, $\ldots$, t. $(c(k_n,m_t),c(l_n,r_t))$.

Hence, we obtain $n\times t$ multiple identities.
The proof is complete.

Example 5.1 Let $Q = \{1,2,3,4\}$. We define the binary operation $\{\cdot\}$.

.	1	2	3	4
1	1	2	4	3
2	3	4	2	1
3	4	1	3	2
4	2	3	1	4

We obtain a quasigroup $(Q_1,\cdot)$ with 3 multiple identities:1 is a $(2,3)$-identity, 3 is a $(3,2)$-identity and 4 is a $(3,3)$-identity.

Let $Q = \{1,2,3,4\}$. We define the binary operation $\{\circ\}$.

$\circ$	1	2	3	4
1	1	2	4	3
2	3	1	2	4
3	2	4	3	1
4	4	3	1	2

We obtain a quasigroup $(Q_2,\circ)$ with 2 multiple identities: 1 is a $(2,2)$-identity and 3 is a $(3,2)$-identity. We will examine the direct product of quasigroups $G = Q_1 \times Q_2$.

$(*)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	4	3	5	6	8	7	13	14	16	15	9	10	12	11
2	3	1	2	4	7	5	6	8	15	13	14	16	11	9	10	12
3	2	4	3	1	6	8	7	5	14	16	15	13	10	12	11	9
4	4	3	1	2	8	7	5	6	16	15	13	14	12	11	9	10
5	9	10	12	11	13	14	16	15	5	6	8	7	1	2	4	3
6	11	9	10	12	15	13	14	16	7	5	6	8	3	1	2	4
7	10	12	11	9	14	16	15	13	6	8	7	5	2	4	3	1
8	12	11	9	10	16	15	13	14	8	7	5	6	4	3	1	2
9	13	14	16	15	1	2	4	3	9	10	12	11	5	6	8	7
10	15	13	14	16	3	1	2	4	11	9	10	12	7	5	6	7
11	14	16	15	13	2	4	3	1	10	12	11	9	6	8	7	5
12	16	15	13	14	4	3	1	2	12	11	9	10	8	7	5	6
13	5	6	8	7	9	10	12	11	1	2	4	3	13	14	16	15
14	7	5	6	8	11	9	10	12	3	1	2	4	15	13	14	16
15	6	8	7	5	10	12	11	9	2	4	3	1	14	16	15	13
16	8	7	5	6	12	11	9	10	4	3	1	2	16	15	13	14

In this case G is a quasigroup with 6 multiple identities: 1 is a $(2,6)$-identity, 3 is a $(6,6)$-identity, 9 is a $(6,2)$-identity, 11 is a $(3,2)$-identity, 13 is a $(6,6)$-identity and 15 is a $(3,6)$-identity. Quasigroup G contain 4 subquasigroups $G_1 = \{1,2,3,4\}$, $G_2 = \{9,10,11,12\}$, $G_3 = \{13,14,15,16\}$, $G_4 = \{1,5,9,13\}$.

6. Some remarks on Haar Measures on paramedial topological quasigroups

By $B(X)$ denote the family of Borel subsets of the space X. A non-negative real-valued function μ defined on the family $B(X)$ of Borel subsets of a space X is said to be a Radon measure on X if it has the following properties:

- $\mu(H) = \sup\{\mu(F) : F \subseteq H, \text{F is a compact subset of H}\}$ for every $H \in B(H)$;
- for every point $x \in X$ there exists an open subset V_x such that $x \in V_x$ and $\mu(V_x) < \infty$.

Definition 6.1 *Let $(A,\cdot)$ be a topological quasigroup with the divisions $\{r,l\}$. A Radon measure μ on A is called:*

– *a **left invariant Haar measure** if $\mu(U) > 0$ and $\mu(xH) = \mu(H)$ for every non-empty open set $U \subseteq A$, a point $x \in A$ and a Borel set $H \in B(A)$;*

– *a **right invariant measure** if $\mu(U) > 0$ and $\mu(Hx) = \mu(H)$ for every non-empty open set $U \subseteq A$, a point $x \in A$ and a Borel set $H \in B(A)$;*

– *an* ***invariant Haar measure*** *if* $\mu(U)>0$ *and* $\mu(xH)=\mu(Hx)=\mu(l(x,H))=\mu(r(H,x))=\mu(H)$ *for every non-empty open set* $U\subseteq A$*, a point* $x\in A$ *and a Borel set* $H\in B(A)$.

Definition 6.2 *We say that on a topological quasigroup* $(A,\cdot)$ ***there exists a unique left (right) invariant Haar measure****, if for every two left (right) invariant Haar measures* μ_1,μ_2 *on* A *there exists a constant* $c>0$ *such that* $\mu_2(H)=c\cdot\mu_1(H)$ *for every Borel set* $H\in B(A)$.

If $(G,+)$ is a locally compact commutative group, then on G there exists a unique invariant Haar measure μ_G [8,14]. We consider on the Abelian topological group $(G,+)$ the invariant measure μ_G. Using the method of proof from [1] we can prove the following Theorems.

Theorem 6.1 *Let* $(G,\cdot)$ *be a locally compact paramedial quasigroup. Then:*

1. There is a commutative topological group $(G,+)$, $\varphi,\psi:G\to G$ *continuous automorphism of* $(G,+)$, $b\in G$, $\varphi^2=\psi^2$ *and* $(G,+,\varphi,\psi,0,b)$*;*

2. If on the Abelian topological group $(G,+)$ *consider the invariant Haar measure* μ_G*, then on* $(G,\cdot)$ *the right(left) invariant Haar measure is unique;*

3. If μ *is a left (right) measure on* $(G,\cdot)$*, then* μ *is a left (right) invariant Haar measure on* $(G,+)$ *too;*

4. On $(G,\cdot)$ *there exists some right invariant Haar measure if and only if* $\mu_G(\varphi(H))=\mu_G(H)$ *for every* $H\in B(A)$*;*

5. If $n<+\infty$*, and on* G *there exists some* $(n,+\infty)$*-identity, then on* $(G,\cdot)$ *the measure* μ_G *is a unique right invariant Haar measure;*

6. If $m<+\infty$*, and on* G *there exists some* $(+\infty,m)$*-identity, then on* $(G,\cdot)$ *the measure* μ_G *is a unique left invariant Haar measure;*

7. If $n,m<+\infty$*, and on* G *there exists some* (n,m)*-identity, then on* $(G,\cdot)$ *the measure* μ_G *is a unique invariant Haar measure.*

Corollary 6.1 *Let* $(G,\cdot)$ *be paramedial quasigroup. Then there is an Abelian group* $(G,+)$ *and element* $q\in Q$ *and group automorphisms* α,β*, such that* $x\cdot y=\alpha(x)+\beta(y)+q$ *for all* $x,y\in Q$ *and* $\alpha\alpha=\beta\beta$ *is fulfilled.*

The Corollary 6.1 was proved in [9],[10],[11],[12].

Theorem 6.2 *Let* $(G,+)$ *be a topological paramedial quasigroup and* $(G,\cdot)$ *be an* (n,m)*-homogeneous isotope of* $(G,+)$*. Then:*

1. On $(G,+)$ *there exists a left (right) invariant Haar measure if and only if on* $(G,\cdot)$ *there exists a left (right) invariant Haar measure.*

2. If on $(G,+)$ *the a left (right) invariant Haar measure is unique, then on* $(G,\cdot)$ *the a left (right) invariant Haar measure is unique too.*

Theorem 6.3. *On a compact paramedial quasigroup* G *there exists a unique Haar measure* μ *for which* $\mu(G)=1$.

The Theorems 6.1, 6.2 and 6.3 for topological medial quasigroups was proved in [1].

7. Examples

Example 7.1 Let $(R,+)$ be a topological Abelian group of real numbers.

1. If $\varphi(x)=x$, $\psi(x)=x$ and $x\cdot y=x+y$, then $(R,\cdot)=g(R,+,\varphi,\psi)$ is a commutative locally compact paramedial quasigroup. By virtue of Theorem 6.1, there exists a left and a right invariant Haar measure on $(R,\cdot)$.

2. If $\varphi(x) = 7x$, $\psi(x) = 7x$ and $x \cdot y = 7x + 7y$, then $(R,\cdot) = g(R,+,\varphi,\psi)$ is a commutative locally compact paramedial quasigroup and on $(R,\cdot)$ as above, by virtue of Theorem 6.1, does not exist any left or right invariant Haar measure.

Example 7.2 Denote by $Z_p = Z / pZ = \{0,1,..., p-1\}$ the cyclic Abelian group of order p. Consider the commutative group $(G,+) = (Z_7,+)$, $\varphi(x) = 4x$, $\psi(x) = 3x$, $x \cdot y = 4x + 3y$.

Then $(G,\cdot) = g(G,+,\varphi,\psi)$ is a medial, paramedial and bicommutative quasigroup with one element $(6,3)$-identity in $(G,\cdot)$, which coincides with the zero element in $(G,+)$.

Example 7.3 Consider the commutative group $(G,+) = (Z_7,+)$, $\varphi(x) = 5x$, $\psi(x) = 3x$ and $x \cdot y = 5x + 3y$. Then $(G,\cdot) = g(G,+,\varphi,\psi)$ is a medial and hexagonal quasigroup where each element is a $(6,6)$-identity in $(G,\cdot)$.

Example 7.4 Consider the commutative group $(G,+) = (Z_5,+)$, $\varphi(x) = x$, $\psi(x) = 4x$ and $x \cdot y = x + 4y$. Then $(G,\cdot) = g(G,+,\varphi,\psi)$ is a medial, paramedial and bicommutative quasigroup where one element is a $(2,1)$- identity in $(G,\cdot)$, which coincides with the zero in $(G,+)$.

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