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ON SOME GROUPS RELATED TO MIDDLE BOL LOOPS

Parascovia SÂRBU, Ion GRECU
Department of Fundamental Mathematics, MSU

A loop $(Q,\cdot)$ is called a middle Bol loop if every loop isotope of $(Q,\cdot)$ satisfies the identity $(x\cdot y)^{-1}=y^{-1}x^{-1}$ (i.e. if the anti-autmorphic inverse property is universal in $(Q,\cdot)$) [6]. The present work continues the investigations from [1] and [5]. Middle Bol loops are isostrophes of left (right) Bol loops [7]. Invariants under this isostrophy are studied and connections between the groups of regular mappings, respectively between the groups of pseudoauto-morphisms (left, right, middle), of a middle Bol loop and those of the corresponding right Bol loop are described in this article. A necessary and sufficient condition when the quotient loops of a middle Bol loop and of the corresponding right Bol loop are isomorphic is given.
Keywords: *Bol loop, middle Bol loop, isostrophes, isostrophy, invariants, universal properties, pseudoautomorphisms.*

ASUPRA UNOR GRUPURI CONEXE BUCLELOR MEDII BOL
O buclă $(Q,\cdot)$ se numeşte buclă medie Bol, dacă orice buclă izotopă cu $(Q,\cdot)$ verifică identitatea $(x\cdot y)^{-1}=y^{-1}x^{-1}$ (adică, dacă proprietatea antiautomorfică de inversabilitate este universală în $(Q,\cdot)$) [6]. În prezentul articol sunt con-tinuate investigaţiile din [1] şi [5]. Buclele medii Bol sunt izostrofi ai buclelor Bol la stânga (la dreapta) [7]. Sunt studiaţi invarianţii la această izostrofie, fiind descrise conexiuni dintre grupurile de substituţii regulare, respectiv dintre grupurile de pseudoautomorfisme (stângi, drepte, medii) ale unei bucle medii Bol şi cele ale buclei Bol la dreapta corespunzătoare ei. De asemenea, este dată o condiţie necesară şi suficientă ca buclele factor, obţinute la factorizarea unei bucle medii Bol şi a buclei Bol la dreapta corespunzătoare ei, să fie izomorfe.
Cuvinte-cheie: *buclă Bol, buclă medie Bol, izostrof, invarianţi, proprietăţi universale, pseudoautomorfisme.*

A loop $(Q,\cdot)$ is called a right Bol loop if it satisfies the identity $(zx\cdot y)x= z(xy\cdot x)$. The class of right Bol loops is one of the well studied in the theory of loops. A loop $(Q,\cdot)$ is called a middle Bol loop if the identity $(x\cdot y)^{-1}=y^{-1}x^{-1}$ (the anti-automorphic inverse property) is universal in $(Q,\cdot)$, i.e. is invariant under loop isotopy. It is shown in [6] that a loop $(Q,\cdot)$ is middle Bol if and only if the corresponding primitive loop $(Q,\cdot,/,\backslash)$ satisfies the identity

$$x(yz\backslash x)=(x/z)(y\backslash x). \qquad (1)$$

A. Gwaramija proved in [4] that middle Bol loops are isostrophs of right (left) Bol loops: a loop $(Q,\circ)$ is middle Bol if and only if there exists a right Bol loop $(Q,\cdot)$ such that

$$x\circ y=(y\cdot xy^{-1})y, \qquad (2)$$

or, equivalently,

$$x\circ y=y^{-1}\backslash x, \qquad (3)$$

for every $x,y\in Q$. So, middle Bol loops are isostrophes of right Bol loops. Invariants under this isostrophy are studied in the present work. Connections between the groups of regular mappings (left, right, middle) of a middle Bol loop and those of the corresponding right Bol loop are established. It is proved that middle pseudoautomorphisms of an arbitrary loop form a group and connections between the groups of pseudoauto-morphisms (left, right, middle) of a middle Bol loop and of the corresponding right Bol loop are proved. It is shown in [1] that if θ is a normal congruence of a right Bol loop $(Q,\cdot)$ then θ is a normal congruence of the corresponding middle Bol loop as well. The converse is proved in the present article and a necessary and sufficient condition when the quotient loops (by some normal congruence) of a middle Bol loop and of the corresponding right Bol loop are isomorphic is given.

From (3) follows

$$x\cdot y=y//x^{-1}, \qquad (4)$$

$\forall x, y \in Q$, where "//" is the right division in $(Q,\circ)$. It is shown in [1], that two middle Bol loops are isotopic (isomorphic) if and only if the corresponding right Bol loops are isotopic (isomorphic). There exists an analogous correspondence between middle Bol loops and left Bol loops [5]. More, if $(Q,\circ)$ is a middle Bol loop and $(Q,\cdot)$ is the corresponding right Bol loop, then $(Q,*)$, where $x*y = y\cdot x$, $\forall x, y \in Q$, is the corresponding left Bol loop for $(Q,\circ)$.

Proposition 1. *Every middle Bol loop* $(Q,\circ)$ *satisfies the equality*

$$(y^n // x^{-1})\circ(x \backslash\backslash y^m) = y^{n+m} , \tag{5}$$

for every $x, y \in Q$ and for every $m,n \in \mathrm{Z}$.

Proof. Let $(Q,\circ)$ be a middle Bol loop and let $(Q,\cdot)$ be the corresponding right Bol loop. According to [4], the loop $(Q,\cdot)$ satisfies the equality

$$xy^m \cdot y^n = x \cdot y^{m+n} , \tag{6}$$

for $\forall x, y \in Q$ and for $\forall m,n \in \mathrm{Z}$. Using (4), the equality (6) implies

$$[y^n //(y^m // x^{-1})^{-1}]\circ x^{-1} = y^{m+n} , \tag{7}$$

where "//" is the left division in $(Q,\circ)$. Now, taking $y^m // x^{-1} = z$ in (7), we get (5). □

Corollary. *If* $(Q,\circ)$ *is a middle Bol loop then, for every* $\forall x, y \in Q$, *and for every* $\forall m,n \in \mathrm{Z}$, *the following equalities hold:*

1) $y = (y^n // x^{-1})\circ(x \backslash\backslash y^{-n+1})$;
2) $y^{n-1} = (y^n // x^{-1})\circ(x \backslash\backslash y^{-1})$;
3) $y^m \circ y^n = y^{m+n}$.

Proof. Taking $m = -n+1$, from (5) get 1). The equality 2) follows from (5), for $m = -1$. Finally, taking $x = e$ (e is the common unit of $(Q,\circ)$ and $(Q,\cdot)$), from (5) follows 3). □

Remark. 1. The equality 3) in the last corollary shows that middle Bol loops are power associative, i.e. each element of $(Q,\circ)$ generates an associative subloop.

2. If $(Q,\cdot)$ is an arbitrary loop then will denote by $R_x^{(\cdot)}$ ($L_x^{(\cdot)}$) the right (respectively, left) translation with the element x in $(Q,\cdot)$, by $\mathcal{L}_{(\cdot)}$ (respectively, $\mathcal{R}_{(\cdot)}, \mathcal{F}_{(\cdot)}$) the group of left (respectively, right, middle) regular mappings of $(Q,\cdot)$, and by N_l (respectively, N_r, N_m) the left (respectively, right, middle) nucleus of $(Q,\cdot)$.

Proposition 2. *If* $(Q,\circ)$ *is a middle Bol loop and* $(Q,\cdot)$ *is the corresponding right Bol loop then the following equalities hold:*

1) $\mathcal{R}_{(\cdot)} = \mathcal{F}_{(\cdot)} = \mathcal{L}_{(\circ)}$;

2) $\mathcal{L}_{(\cdot)} = \mathcal{F}_{(\circ)}$;

3) $\mathcal{F}^*_{(\cdot)} = \mathcal{R}_{(\circ)}$,

where $\mathcal{F}^*_{(\cdot)}$ *is the group of conjugates of the middle regular mappings of* $(Q,\cdot)$.

Proof. 1) Let $\rho \in \mathcal{R}_{(\cdot)}$. Then $\rho(x\cdot y) = x\cdot\rho(y) \Leftrightarrow \rho(y // x^{-1}) = \rho(y) // x^{-1} \Leftrightarrow \rho(y) = \rho(y // x^{-1})\circ x^{-1} \Leftrightarrow \rho(y) = \rho(y // x)\circ x$, $\forall x, y \in Q$, where "//" is the left division in $(Q,\circ)$. Denoting $y // x$ by z, the last equality take the form: $\rho(z\circ x) = \rho(z)\circ x$, $\forall z, x \in Q$, i.e. $\rho \in \mathcal{L}_{(\circ)}$, so $\mathcal{R}_{(\cdot)} = \mathcal{L}_{(\circ)}$. Analogously, if $\varphi \in \mathcal{F}_{(\cdot)}$ then $\exists \varphi^* \in S_Q$ such that $\varphi(x)\cdot y = x\cdot\varphi^*(y) \Leftrightarrow y // I\varphi(x) = \varphi^*(y) // x^{-1} \Leftrightarrow \varphi^*(y) = (y // I\varphi(x))\circ x^{-1}$, $\forall x, y \in Q$. Denoting $y // I\varphi(x)$ by z in the last equality, get: $\varphi^*(z\circ I\varphi(x)) = z\circ x^{-1}$, which is equivalent to

$$\varphi^*(z\circ x) = z\circ I\varphi^{-1}I(x) . \tag{8}$$

Taking $z=e$ in (8), where e is the common unit of the loops $(Q,\circ)$ and $(Q,\cdot)$, have $\varphi^*(x)=I\varphi^{-1}I(x)$, $\forall x\in Q$, i.e. $\varphi^*=I\varphi^{-1}I$, so using (8) get: $I\varphi^{-1}I(z\circ x)=z\circ I\varphi^{-1}I(x)\Leftrightarrow z\circ x=I\varphi I(z\circ I\varphi^{-1}I(x))\Leftrightarrow z\circ I\varphi I(x)=I\varphi I(z\circ x)\Leftrightarrow z\circ I\varphi(x^{-1})=I\varphi(x^{-1}\circ z^{-1})\Leftrightarrow \varphi(x^{-1})\circ z^{-1}=\varphi(x^{-1}\circ z^{-1})$, $\forall x,z\in Q$, which is equivalent to $\varphi\in\mathcal{L}_{(\circ)}$, hence $\mathcal{F}_{(\cdot)}=\mathcal{L}_{(\circ)}$.

2) $\lambda\in\mathcal{L}_{(\cdot)}\Leftrightarrow\lambda(x\cdot y)=\lambda(x)\cdot y\Leftrightarrow\lambda(y//x^{-1})=y//\lambda(x)^{-1}$. Denoting $y//x^{-1}$ by z, i.e. $y=z\circ x^{-1}$, have: $\lambda(y//x^{-1})=y//\lambda(x)^{-1}\Leftrightarrow\lambda(z)=(z\circ x^{-1})//I\lambda(x)\Leftrightarrow z\circ x^{-1}=\lambda(z)\circ I\lambda(x)\Leftrightarrow z\circ I\lambda^{-1}I(x)=\lambda(z)\circ x$, $\forall x,z\in Q$, i.e. $\lambda\in\mathcal{F}_{(\circ)}$, hence $\mathcal{L}_{(\cdot)}=\mathcal{F}_{(\circ)}$.

3) $\varphi^*\in\mathcal{F}^*_{(\cdot)}$ if and only if $\exists\varphi\in\mathcal{F}_{(\cdot)}$: $\varphi(x)\cdot y=x\cdot\varphi^*(y)\Leftrightarrow y//(\varphi(x))^{-1}=\varphi^*(y)//x^{-1}\Leftrightarrow\varphi^*(y)=(y//I\varphi(x))\circ x^{-1}$. Denoting $y//I\varphi(x)$ by z, i.e. $y=z\circ I\varphi(x)$, get: $\varphi^*(z\circ I\varphi(x))=z\circ x^{-1}$, which is equivalent to

$$\varphi^*(z\circ x)=z\circ I\varphi^{-1}I(x). \tag{9}$$

The last equality, for $z=e$, gives $\varphi^*(x)=I\varphi^{-1}I(x)$, for every $x\in Q$, i.e. $\varphi^*=I\varphi^{-1}I$, so (9) takes the form:

$$\varphi^*(z\circ x)=z\circ\varphi^*(x),$$

$\forall x,z\in Q$, i.e. $\varphi^*\in\mathcal{R}_{(\circ)}$, hence $\mathcal{F}^*_{(\cdot)}=\mathcal{R}_{(\circ)}$. □

Corollary. *If $(Q,\circ)$ is a middle Bol loop and $(Q,\cdot)$ is the corresponding right Bol loop then the following equalities hold:*

a) $\mathcal{L}_{(\circ)}=I\mathcal{R}_{(\circ)}I$,

b) $N_r^{(\cdot)}=N_m^{(\cdot)}=N_l^{(\circ)}=N_r^{(\circ)}$,

c) $N_l^{(\cdot)}=N_m^{(\circ)}$.

Proof. a) $\lambda\in\mathcal{L}_{(\circ)}\Leftrightarrow\lambda(x\circ y)=\lambda(x)\circ y\Leftrightarrow I\lambda(x\circ y)=Iy\circ I\lambda(x)\Leftrightarrow I\lambda I(y^{-1}\circ x^{-1})=y^{-1}\circ I\lambda I(x^{-1})$, $\forall x,y\in Q$, so $\lambda\in\mathcal{L}_{(\circ)}\Leftrightarrow I\lambda I\in\mathcal{R}_{(\circ)}\Leftrightarrow\lambda\in I\mathcal{R}_{(\circ)}I$, i.e. $\mathcal{L}_{(\circ)}=I\mathcal{R}_{(\circ)}I$.

b), c): It is known that in an arbitrary loop $(Q,\cdot)$: $N_l^{(\cdot)}=\mathcal{L}_{(\cdot)}(e)$, $N_r^{(\cdot)}=\mathcal{R}_{(\cdot)}(e)$ and $N_m^{(\cdot)}=\mathcal{F}_{(\cdot)}(e)=\mathcal{F}^*_{(\cdot)}(e)$, where e is the unit of $(Q,\cdot)$. So, the equalities b) and c) follow from Proposition 2. □

Let $(Q,\cdot)$ be an arbitrary loop, $\varphi\in S_Q$ and $c\in Q$. Remind that: a) φ is called a *left (resp. right) pseudoautomorphism* of $(Q,\cdot)$, with the companion c, c if the equality

$$c\cdot\varphi(x\cdot y)=[c\cdot\varphi(x)]\cdot\varphi(y)\ \text{(resp., } \varphi(x\cdot y)\cdot c=\varphi(x)\cdot[\varphi(y)\cdot c]\text{)}$$

holds, for every $x,y\in Q$. b) φ is called a *middle pseudoautomorphism* with the companion c if the equality

$$\varphi(x\cdot y)=[\varphi(x)/c^{-1}]\cdot[c\backslash\varphi(y)], \tag{10}$$

holds, for every $x,y\in Q$, where c^{-1} is the right inverse of c. Middle pseudoautomorphisms of (right) Bruck loops are studied in [2]. Below we denote by $PS_r^{(\cdot)}$ (respectively, $PS_l^{(\cdot)}$, $PS_m^{(\cdot)}$) the set of all right pseudoautomorphisms (resp. left, middle pseudoautomorphisms) of the loop $(Q,\cdot)$.

Lemma 1. *If $(Q,\cdot)$ is an arbitrary loop with the unit 1,then $\varphi(1)=1$, for every $\varphi\in PS_m^{(\cdot)}$.*

Proof. Taking $x=1$ in (10), get $\varphi(y)=(\varphi(1)/(c\backslash 1))\cdot(c\backslash\varphi(y))$, for every $\forall y\in Q$. Now, making the substitution $y\to\varphi^{-1}(y)$ in the last equality, have $y=(\varphi(1)/(c\backslash 1))\cdot(c\backslash y)$, for $\forall y\in Q$, which implies (for $y=1$): $1=(\varphi(1)/(c\backslash 1))\cdot(c\backslash 1)$, so $\varphi(1)=1$. □

Proposition 3. *If φ is a middle pseudoautomorphism of a loop $(Q,\cdot)$, with the companion c, then φ^{-1} is a middle pseudoautomorphism of $(Q,\cdot)$, with the companion $c_1 = \varphi^{-1}(c \backslash 1)$.*

Proof. Let $\varphi \in PS_m^{(\cdot)}$ with the companion c. Making the substitutions $x = \varphi^{-1}(x)$ and $y = \varphi^{-1}(y)$ in (10), get

$$\varphi(\varphi^{-1}(x) \cdot \varphi^{-1}(y)) = (x/(c \backslash 1)) \cdot (c \backslash y),$$

which is equivalent to

$$\varphi^{-1}(x) \cdot \varphi^{-1}(y) = \varphi^{-1}(R_{c\backslash 1}^{-1}(x) \cdot L_c^{-1}(y)),$$

for $\forall x, y \in Q$. Now, replacing $x \to R_{c\backslash 1}(x)$ and $y \to L_c(y)$ in the last equality, have

$$\varphi^{-1}R_{c\backslash 1}(x) \cdot \varphi^{-1}L_c(y) = \varphi^{-1}(x \cdot y), \tag{11}$$

for $\forall x, y \in Q$. Taking $y = 1$ in (11), get

$$\varphi^{-1}R_{c\backslash 1}(x) \cdot \varphi^{-1}(c) = \varphi^{-1}(x),$$

which is equivalent to

$$\varphi^{-1}R_{c\backslash 1}(x) = \varphi^{-1}(x)/\varphi^{-1}(c), \tag{12}$$

for $\forall x \in Q$. Also, taking $x = y = 1$ in (11), have $\varphi^{-1}(c \backslash 1) \cdot \varphi^{-1}(c) = 1$, i.e.

$$\varphi^{-1}(c) = \varphi^{-1}(c \backslash 1) \backslash 1. \tag{13}$$

Using (13), from (12) follows:

$$\varphi^{-1}R_{c\backslash 1}(x) = \varphi^{-1}(x)/[\varphi^{-1}(c \backslash 1) \backslash 1], \tag{14}$$

for $\forall x \in Q$. For $x = 1$, the equality (11) implies $\varphi^{-1}R_{c\backslash 1}(1) \cdot \varphi^{-1}L_c(y) = \varphi^{-1}(y)$, i.e.

$$\varphi^{-1}L_c(y) = \varphi^{-1}(c \backslash 1) \backslash \varphi^{-1}(y), \tag{15}$$

for $\forall y \in Q$. Using (14) and (15) in (11), get

$$[\varphi^{-1}(x)/(\varphi^{-1}(c \backslash 1) \backslash 1)] \cdot [\varphi^{-1}(c \backslash 1) \backslash \varphi^{-1}(y)] = \varphi^{-1}(x \cdot y)$$

or, denoting in the last equality $\varphi^{-1}(c \backslash 1) = c_1$:

$$[\varphi^{-1}(x)/(c_1 \backslash 1)] \cdot [c_1 \backslash \varphi^{-1}(y)] = \varphi^{-1}(x \cdot y),$$

i.e. $\varphi^{-1} \in PS_m^{(\cdot)}$, with the companion $c_1 = \varphi^{-1}(c \backslash 1)$. □

Proposition 4. *If φ and ψ are two middle pseudoautomorphisms of a loop $(Q,\cdot)$, with the companions c and b, respectively, then $\varphi\psi$ is a middle pseudoautomorphisms of $(Q,\cdot)$, with the companion $c_2 = R_{c\backslash 1}^{-1}\, \varphi\, R_{b\backslash 1}^{-1}(1)$.*

Proof. Let $\varphi, \psi \in PS_m^{(\cdot)}$, with the companions c and b, respectively. We" ll use the equivalence:

$\varphi \in PS_m^{(\cdot)}$**,** with the companion c, if and only if $(R_{c\backslash 1}^{-1}\, \varphi, L_c^{-1}\varphi, \varphi) \in A_{(\cdot)}$**,**

and the analogous equivalence for ψ, where $A_{(\cdot)}$ is the group of autotopisms of the loop $(Q,\cdot)$.

So, $\varphi, \psi \in PS_m^{(\cdot)}$ if and only if $(R_{c\backslash 1}^{-1}\, \varphi\, R_{b\backslash 1}^{-1}\, \psi, L_c^{-1}\varphi L_b^{-1}\psi, \varphi\psi) \in A_{(\cdot)}$, i.e.

$$\varphi\psi(x \cdot y) = R_{c\backslash 1}^{-1}\, \varphi\, R_{b\backslash 1}^{-1}\, \psi(x) \cdot L_c^{-1}\varphi L_b^{-1}\psi(y), \tag{16}$$

for $\forall x, y \in Q$. Taking $y = 1$ in (16) get: $\varphi\psi(x) = R_{c\backslash 1}^{-1}\, \varphi\, R_{b\backslash 1}^{-1}\, \psi(x) \cdot L_c^{-1}\varphi L_b^{-1}(1)$ which is equivalent to

$$R_{c\backslash 1}^{-1}\, \varphi\, R_{b\backslash 1}^{-1}\, \psi(x) = \varphi\psi(x)/L_c^{-1}\varphi L_b^{-1}(1), \tag{17}$$

for $\forall x \in Q$. Now, taking $x = y = 1$ in (16) have $1 = R_{c\backslash 1}^{-1}\, \varphi\, R_{b\backslash 1}^{-1}(1) \cdot L_c^{-1}\varphi L_b^{-1}(1)$, i.e.

$$L_c^{-1}\varphi L_b^{-1}(1) = R_{c\backslash 1}^{-1}\, \varphi\, R_{b\backslash 1}^{-1}(1) \backslash 1. \tag{18}$$

Using (18) in (17) get

$$R_{c\backslash 1}^{-1}\, \varphi\, R_{b\backslash 1}^{-1}\, \psi(x) = \varphi\psi(x)/[R_{c\backslash 1}^{-1}\, \varphi\, R_{b\backslash 1}^{-1}(1) \backslash 1], \tag{19}$$

for $\forall x \in Q$. Taking $x=1$ in (16) have $\varphi\psi(y) = R_{c\backslash 1}^{-1}\ \varphi\ R_{b\backslash 1}^{-1}(1)\cdot L_c^{-1}\varphi L_b^{-1}\psi(y)$, which is equivalent to

$$L_c^{-1}\varphi L_b^{-1}\psi(y) = R_{c\backslash 1}^{-1}\ \varphi\ R_{b\backslash 1}^{-1}(1)\backslash \varphi\psi(y), \qquad (20)$$

for $\forall y \in Q$. Using (19) and (20) in (16), get

$$\varphi\psi(x\cdot y) = [\varphi\psi(x)/[R_{c\backslash 1}^{-1}\ \varphi\ R_{b\backslash 1}^{-1}(1)\backslash 1]]\cdot[R_{c\backslash 1}^{-1}\ \varphi\ R_{b\backslash 1}^{-1}(1)\backslash \varphi\psi(y)],$$

for $\forall x,y\in Q$. Now, denoting $R_{c\backslash 1}^{-1}\ \varphi\ R_{b\backslash 1}^{-1}(1) = c_2$, the last equality takes the form

$$\varphi\psi(x\cdot y) = [\varphi\psi(x)/(c_2\backslash 1)]\cdot[c_2\backslash \varphi\psi(y)],$$

for $\forall x,y\in Q$, i.e. $\varphi\psi \in PS_m^{(\cdot)}$, with the companion $c_2 = R_{c\backslash 1}^{-1}\ \varphi\ R_{b\backslash 1}^{-1}(1)$. □

Corollary. The set of all middle pseudoautomorphisms of a loop forms a group.

Proposition 5. *Let $(Q,\cdot)$ be a right Bol loop and let $(Q,\circ)$ be the corresponding middle Bol loop. The following sentences are true:*

1. $PS_m^{(\cdot)} = PS_r^{(\circ)}$; 2. $PS_l^{(\cdot)} = PS_m^{(\circ)}$;
3. $PS_r^{(\cdot)} = PS_r^{(\circ)}$; 4. $\alpha \in PS_l^{(\circ)} \Leftrightarrow I\alpha I \in PS_r^{(\circ)}$.

Proof. 1. Let φ be a middle pseudoautomorphism of $(Q,\cdot)$ with the companion c. Then φ satisfies (10). Using (4), the equality (10) takes the form:

$$\varphi(y//x^{-1}) = [\varphi(y)\circ c^{-1}]//[c//\varphi(x)^{-1}]^{-1}$$

which is equivalent to

$$\varphi(y)\circ c^{-1} = \varphi(y//x^{-1})\circ[c//\varphi(x)^{-1}]^{-1}, \qquad (21)$$

where $(//)$ is the left division in $(Q,\circ)$. So as $(Q,\circ)$ is a middle Bol loop, $[c//\varphi(x)^{-1}]^{-1} = \varphi(x)\backslash\backslash c^{-1}$, $\forall x\in Q$, where "$\backslash\backslash$" is the right division in $(Q,\circ)$, hence (21) is equivalent to

$$\varphi(y)\circ c^{-1} = \varphi(y//x^{-1})\circ(\varphi(x)\backslash\backslash c^{-1}).$$

Denoting $y//x^{-1}$ by z, the last equality takes the form:

$$\varphi(z\circ x^{-1})\circ c^{-1} = \varphi(z)\circ(\varphi(x)\backslash\backslash c^{-1}). \qquad (22)$$

Remark that, for $z=x=e$, from (22) follows $\varphi(e)=e$. Taking $z=e$ in (22) and using the equality $\varphi(e)=e$, get $\varphi(x)\backslash\backslash c^{-1} = \varphi(x^{-1})\circ c^{-1}$, so (22) is equivalent to

$$\varphi(z\circ x^{-1})\circ c^{-1} = \varphi(z)\circ(\varphi(x^{-1})\circ c^{-1}),$$

i.e. φ is a right pseudoautomorphism $(Q,\circ)$ with the companion c^{-1}.

Conversely, $\varphi\in PS_r^{(\circ)} \Rightarrow \exists c\in Q$: $\varphi(x\circ y)\circ c = \varphi(x)\circ(\varphi(y)\circ c)$, $\forall x,y\in Q$. Using (3), from the last equality get

$$c^{-1}\backslash\varphi(y^{-1}\backslash x) = (c^{-1}\backslash\varphi(y))^{-1}\backslash\varphi(x) \Leftrightarrow \varphi(x) = I(c^{-1}\backslash\varphi(y))\cdot(c^{-1}\backslash\varphi(y^{-1}\backslash x)).$$

Denoting $y^{-1}\backslash x = z$, the last equality implies

$$\varphi(y^{-1}\cdot z) = I(c^{-1}\backslash\varphi(y))\cdot(c^{-1}\backslash\varphi(z)). \qquad (23)$$

Taking $z=e$, from (23) follows $\varphi(y^{-1}) = I(c^{-1}\backslash\varphi(y))\cdot c$, hence

$$\varphi(y^{-1})\cdot c^{-1} = I(c^{-1}\backslash\varphi(y)). \qquad (24)$$

Using (24), the equality (23) implies $\varphi(y^{-1}\cdot z) = (\varphi(y^{-1})\cdot c^{-1})\cdot(c^{-1}\backslash\varphi(z))$, hence

$$\varphi(y^{-1}\cdot z) = (\varphi(y^{-1})/c)\cdot(c^{-1}\backslash\varphi(z)),$$

$\forall y,z\in Q$, i.e. φ is a middle pseudoautomorphism of $(Q,\cdot)$, with the companion c^{-1}.

2. Let φ be a left pseudoautomorphism of the loop $(Q,\cdot)$, with the companion b:

$$b\cdot\varphi(x\cdot y) = (b\cdot\varphi(x))\cdot\varphi(y), \qquad (25)$$

$\forall x, y \in Q$. Using (4) in (25), get:

$$\varphi(y // x^{-1}) // b^{-1} = \varphi(y) // (\varphi(x) // b^{-1})^{-1}. \quad (26)$$

So as $(Q, \circ)$ satisfies the anti-automorphic inverse property, denoting $[\varphi(x) // b^{-1}]^{-1} = u$, get: $\varphi(x) // b^{-1} = u^{-1} \Leftrightarrow u^{-1} \circ b^{-1} = \varphi(x) \Leftrightarrow b \circ u = \varphi(x)^{-1} \Leftrightarrow u = b \backslash\backslash \varphi(x)^{-1}$, so

$$[\varphi(x) // b^{-1}]^{-1} = b \backslash\backslash \varphi(x)^{-1}. \quad (27)$$

Using (27), the equality (26) takes the form:

$$\varphi(y // x^{-1}) // b^{-1} = \varphi(y) // (b \backslash\backslash \varphi(x)^{-1}). \quad (28)$$

Denoting $y // x^{-1} = z$, i.e. $z \circ x^{-1} = y$, (28) implies

$$\varphi(z) // b^{-1} = \varphi(z \circ x^{-1}) // (b \backslash\backslash \varphi(x)^{-1}),$$

which is equivalent to:

$$\varphi(z \circ x^{-1}) = (\varphi(z) // b^{-1}) \circ (b \backslash\backslash \varphi(x)^{-1}), \quad (29)$$

and, taking $z = e$, the last equality implies

$$\varphi(x^{-1}) = b \circ (b \backslash\backslash \varphi(x)^{-1}) = \varphi(x)^{-1},$$

$\forall x \in Q$, so (29) is equivalent to

$$\varphi(z \circ x) = (\varphi(z) // b^{-1}) \circ (b \backslash\backslash \varphi(x)),$$

$\forall z, x \in Q$, i.e. $\varphi \in PS_m^{(\circ)}$, with the companion b.

Conversely, if $\varphi \in PS_m^{(\circ)}$ then $\exists c \in Q$:

$$\varphi(x \circ y) = [\varphi(x) // c^{-1}] \circ [c \backslash\backslash \varphi(y)], \quad (30)$$

$\forall x, y \in Q.$ Denoting $\varphi(x) // c^{-1} = u$ and using (3), get $u \circ c^{-1} = \varphi(x) \Leftrightarrow c \backslash u = \varphi(x) \Leftrightarrow c \cdot \varphi(x) = u$, so

$$\varphi(x) // c^{-1} = c \cdot \varphi(x). \quad (31)$$

Analogously, denoting $c \backslash\backslash \varphi(y) = v$ and using (4), get: $c \circ v = \varphi(y) \Leftrightarrow v^{-1} \backslash c = \varphi(y) \Leftrightarrow$ $c = v^{-1} \cdot \varphi(y) \Leftrightarrow v^{-1} = c \cdot I\varphi(y) \Leftrightarrow v = I(c \cdot I\varphi(y))$, so

$$\varphi(x) // c^{-1} = I(c \cdot I\varphi(y)), \quad (32)$$

$\forall y \in Q$. Now, using (31) and (32), the equality (30) takes the form:

$$\varphi(y^{-1} \backslash x) = (c \cdot I\varphi(y)) \backslash (c \cdot \varphi(x)),$$

which is equivalent to

$$c \cdot \varphi(x) = (c \cdot I\varphi(y)) \cdot \varphi(y^{-1} \backslash x).$$

Denoting $y^{-1} \backslash x = z$, the last equality implies

$$c \cdot \varphi(y^{-1} \cdot x) = (c \cdot I\varphi(y)) \cdot \varphi(z). \quad (33)$$

Taking $z = e$, from the last equality follows $c \cdot \varphi(y^{-1}) = c \cdot I\varphi(y)$, which implies $\varphi I(y) = I\varphi(y)$, $\forall y \in Q$, so

$$\varphi I = I\varphi. \quad (34)$$

Now, using (33) and (34), get:

$$c \cdot \varphi(y^{-1} \cdot x) = (c \cdot \varphi(y^{-1})) \cdot \varphi(z),$$

$\forall y, z \in Q$, i.e. $\varphi \in PS_l^{(\cdot)}$.

3. Let φ be a right pseudoautomorphism of $(Q, \cdot)$ with the companion b, then

$$\varphi(x \cdot y) \cdot b = \varphi(x) \cdot (\varphi(y) \cdot b),$$

$\forall x, y \in Q$. Using (4), the last equality takes the form: $b // \varphi(y // x^{-1})^{-1} = (b // \varphi(y)^{-1}) // \varphi(x)^{-1}$, which is equivalent to $b // \varphi(y)^{-1} = [b // \varphi(y // x^{-1})^{-1}] \circ \varphi(x)^{-1}$ or, denoting $y // x^{-1} = z$, i.e. $z \circ x^{-1} = y$, to

$$b // \varphi(z \circ x^{-1})^{-1} = [b // \varphi(z)^{-1}] \circ \varphi(x)^{-1}. \quad (35)$$

Taking $z = e$ in (35), have:

$$b // \varphi(x^{-1})^{-1} = b \circ \varphi(x)^{-1}.$$

Now, using the last equality and replacing z by z^{-1} in (35), get

$$b \circ \varphi(x \circ z)^{-1} = b //(\varphi(x \circ z)^{-1})^{-1} = b // \varphi(z^{-1} \circ x^{-1})^{-1} = [b // \varphi(z)^{-1}] \circ \varphi(x)^{-1}.$$

So, using anti-AIP, which is valid in $(Q, \circ)$, get

$$\varphi(x \circ z) \circ b^{-1} = \varphi(x) \circ [\varphi(z) \circ b^{-1}],$$

$\forall x, z \in Q$, i.e. φ is a right pseudoautomorphism of $(Q, \circ)$, with the companion b^{-1}.

4. If $\varphi \in PS_l^{(\circ)}$ then there exists an element $c \in Q$ such that

$$c \circ \varphi(x \circ y) = (c \circ \varphi(x)) \circ \varphi(y), \quad (36)$$

for $\forall x, y \in Q.$ Using (3), the equality (36) takes the form $I\varphi(x) \backslash c = I\varphi(y) \cdot (I\varphi(y^{-1} \backslash x) \backslash c)$, so, denoting $y^{-1} \backslash x = z$, get

$$I\varphi(y^{-1} \cdot z) \backslash c = I\varphi(y) \cdot (I\varphi(z) \backslash c), \quad (37)$$

which implies, for $z = e$ $I\varphi I(y) \backslash c = I\varphi(y) \cdot c \Leftrightarrow I\varphi(y^{-1}) \backslash c = I\varphi(y) \cdot c$, i.e.

$$I\varphi(y) \backslash c = I\varphi I(y) \cdot c. \quad (38)$$

Using (38), from (37) follows $I\varphi I(y^{-1} \cdot z) \cdot c = I\varphi I(y^{-1}) \cdot (I\varphi I(z) \cdot c)$, $\forall y, z \in Q$, i.e. $I\varphi I$ is a right pseudoautomorphism of $(Q, \cdot)$, with the companion c. □

Definition. A loop $(Q, \cdot)$ is said to be a (right) Bruck loop if it satisfies the right Bol identity $(zy \cdot x)y = z(yx \cdot y)$ and the anti-automorphic inverse property (AIP): $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$, $\forall x, y \in Q$.

Proposition 7. *Commutative middle Bol loops are isostrophs of Bruck loops.*

Proof. Let $(Q, \circ)$ be a commutative middle Bol loop and let $(Q, \cdot)$ be the corresponding right Bol loop. Then: $x \circ y = y \circ x \Leftrightarrow y^{-1} \backslash x = x^{-1} \backslash y \Leftrightarrow y = x^{-1} \cdot (y^{-1} \backslash x)$, $\forall x, y \in Q$. Denoting $y^{-1} \backslash x = z$, in the last identity, get: $y = (y^{-1} \cdot z)^{-1} \cdot z \Leftrightarrow y \cdot z^{-1} = (y^{-1} \cdot z)^{-1} \Leftrightarrow y^{-1} \cdot z^{-1} = (y \cdot z)^{-1}$, $\forall y, z \in Q$. □

Proposition 8. *Let $(Q, \circ)$ be a commutative middle Bol loop and let $(Q, \cdot)$ be the corresponding right Bol loop. Then*

1. *If $\varphi \in PS_r^{(\circ)}$ with the companion c, than $c \in N_m^{(\circ)}$;*

2. *If $\varphi \in PS_m^{(\circ)}$ with the companion c, than c^{-1} is a companion of φ as well, and $c^2 \in N_m^{(\circ)}$.*

Proof. It is proved in [7] that if $(Q, \cdot)$ is a (right) Bruck loop, than the following sentences are true: a) if $\varphi \in PS_r^{(\cdot)}$ with the companion c, than $c \in N_l^{(\cdot)}$; b) if $\varphi \in PS_l^{(\cdot)}$ with the companion c, than c^{-1} is a companion of φ as well, and $c^2 \in N_l^{(\cdot)}$. Hence, Proposition 8 follows from the mention results, Proposition 7 and Corollary after Proposition 2. □

Corollary. *Every right pseudoautomorphism of a commutative middle Bol loop $(Q, \circ)$ with a trivial middle nucleus is an automorphism of $(Q, \circ)$.*

Let $(Q, \cdot)$ be a quasigroup. Remind that a binary equivalence relation $\theta \subseteq Q \times Q$ is called a congruence of $(Q, \cdot)$ if

$$x\theta y \Rightarrow c \cdot x\theta c \cdot y \text{ and } x \cdot c\theta y \cdot c, \ \forall c \in Q.$$

A congruence θ of $(Q,\cdot)$ is called a normal congruence if, for $\forall c \in Q$, each of the relations $c \cdot x\theta c \cdot y$ and $x \cdot c\theta y \cdot c$ implies $x\theta y$ [3,6].

Lemma 2. *Let θ be a normal congruence of a middle Bol loop $(Q,\circ)$. Then:*

a) $x\theta y \Rightarrow x^{-1}\theta y^{-1}$

b) $K_x^{-1} = K_{x^{-1}}$, $\forall K_x \in Q/\theta$.

Proof. Let $(Q,\cdot)$ be the corresponding right Bol loop. a) Using (3), have: $x\theta y \Leftrightarrow$ $(x // c^{-1}) \circ c^{-1}\theta (y // c^{-1}) \circ c^{-1}$ $\Leftrightarrow$ $x // c^{-1}\theta y // c^{-1}$ $\Leftrightarrow$ $c \cdot x\theta c \cdot y$, $\forall c \in Q$. According to (3), $x \circ y = y^{-1} \backslash x$ $\Leftrightarrow$ $x = y^{-1} \cdot (x \circ y)$, so $x\theta y$ $\Leftrightarrow$ $e\theta x^{-1} \circ y$ $\Leftrightarrow y^{-1}\theta y^{-1} \cdot (x^{-1} \circ y)$ $\Leftrightarrow y^{-1}\theta x^{-1}$.

b) Let $K_x \in Q/\theta$. Then $K_x^{-1} = \{y^{-1} \mid y\theta x\}$, hence $y \in K_{x^{-1}}$ $\Rightarrow$ $y\theta x^{-1}$ $\Rightarrow$ $y^{-1}\theta x$ $\Rightarrow$ $y^{-1} \in K_x$ $\Rightarrow$ $y \in K_x^{-1} \Rightarrow K_{x^{-1}} \subseteq K_x^{-1}$. Analogously, $y \in K_x^{-1} \Rightarrow y^{-1} \in K_x \Rightarrow y^{-1}\theta x \Rightarrow y\theta x^{-1} \Rightarrow y \in K_{x^{-1}} \Rightarrow$ $K_x^{-1} \subseteq K_{x^{-1}}$. □

Remark. 1. If θ is a normal congruence of a quasigroup $(Q,\cdot)$ then $K_x \backslash K_y = K_{x\backslash y}$ and $K_x / K_y = K_{x/y}$, for every $K_x, K_y \in Q/\theta$. Indeed, $(Q/\theta,\cdot)$ is a quasigroup with $K_x \cdot K_y = K_{x\cdot y}$, $\forall K_x, K_y \in Q/\theta$. Let $K_x, K_y \in Q/\theta$. Denoting $K_x \backslash K_y = K_z$, have: $K_{xz} = K_x K_z = K_y$ i.e. $xz\theta y$. So as $(Q,\cdot)$ is a quasigroup, there exists an element $y' \in Q$ such that $y = x \cdot y'$. Hence $xz\theta y$ $\Leftrightarrow$ $xz\theta xy'$ $\Leftrightarrow$ $z\theta y'$, where $y' = x \backslash y$, i.e. $K_x \backslash K_y = K_z = K_{y'} = K_{x\backslash y}$. Analogously, $K_x / K_y = K_z \Leftrightarrow$ $K_{zy} = K_z K_y = K_x$ $\Leftrightarrow$ $zy\theta x$. Denoting $x = x'y$, get: $zy\theta x$ $\Rightarrow$ $zy\theta x'y$ $\Rightarrow$ $z\theta x' \Rightarrow z\theta(x/y)$ $\Rightarrow$ $K_x / K_y = K_z = K_{x'} = K_{x/y}$. □

2. If θ is a normal congruence of a middle Bol loop $(Q,\circ)$ then $(Q/\theta,\circ)$ is a middle Bol loop as well. Indeed, according to p.1 of this Remark, the primitive loop $(Q/\theta,\circ,//,\backslash\backslash)$, corresponding to $(Q/\theta,\circ)$, satisfies the identity (1). □

It is proved in [1] that if θ is a normal congruence of a right Bol loop $(Q,\cdot)$ then θ is a normal congruence of the corresponding middle Bol loop as well. The converse is proved in the following proposition.

Proposition 9. *If θ is a normal congruence of a middle Bol loop $(Q,\circ)$ then θ is a normal congruence of the corresponding right Bol loop $(Q,\cdot)$ as well.*

Proof. Let consider the canonical projection

$$\pi : (Q,\circ) \rightarrow (Q/\theta,\circ), \pi(x) = K_x,$$

which is a surjective homomorfism of loops. According to the last Remark, $(Q/\theta,\circ)$ is a middle Bol loop. It is clear that $(Q/\theta,\cdot)$ is the corresponding right Bol loop. Using (3) and Lemma 2, have: $K_{y^{-1}\backslash x} = K_{x\circ y} = K_x \circ K_y = K_y^{-1} \backslash K_x = K_{y^{-1}} \backslash K_x \Rightarrow K_{y^{-1}} \cdot K_{y^{-1}\backslash x} = K_x$. Denoting $y^{-1} \backslash x$ by z, get $y^{-1} \cdot z = x$, hence $K_{y^{-1}} \cdot K_z = K_{y^{-1}\cdot z} \Rightarrow K_y \cdot K_z = K_{y\cdot z}$, $\forall y, z \in Q$, so the function

$$\pi : (Q,\cdot) \rightarrow (Q/\theta,\cdot), \pi(x) = K_x,$$

is a homomorphism of quasigroups, from $(Q,\cdot)$ to $(Q/\theta,\cdot)$. The homomorphism π defines on Q the normal congruence θ', where

$$x\theta' y \Leftrightarrow \pi(x) = \pi(y) \Leftrightarrow x\theta y,$$

so $\theta' = \theta$ is a normal congruence of $(Q,\cdot)$. □

Remark. 1. Let $(Q,\circ)$ be a middle Bol loop and let $(Q,\cdot)$ be the corresponding right Bol loop. If θ is a normal congruence of $(Q,\cdot)$ (or $(Q,\circ)$) then $(Q/\theta,\cdot)$ is the right Bol loop corresponding to the middle Bol loop $(Q/\theta,\circ)$. Indeed, denoting by $(Q/\theta,*)$ the corresponding right Bol loop for $(Q/\theta,\circ)$, get:

$$K_x * K_y = K_y // K_{x^{-1}} = K_{y//x^{-1}} = K_{x\cdot y} = K_x \cdot K_y,$$

$\forall K_x, K_y \in Q/\theta$, so $"*" = "\cdot"$.

Proposition 10. *Let* $\varphi:(Q,\cdot)\to(Q,\circ)$ *be a surjective homomorphism of loops, from a middle Bol loop* $(Q,\circ)$ *to the corresponding right Bol loo* $(Q,\cdot)$ *and let* θ *be the normal congruence defined by* φ *in* $(Q,\cdot)$. *If* φ *is semi-admissible with respect to* θ *, then* $(Q/\theta,\cdot)\cong(Q/\theta,\circ)$.

Proof. The binary relation θ defined on Q by the equivalence

$$x\theta y \Leftrightarrow \varphi(x)=\varphi(y),$$

is a normal congruence in $(Q,\cdot)$. Let consider the function

$$\varphi':(Q/\theta,\cdot)\to(Q/\theta,\circ), \varphi'(K_x)=K_{\varphi(x)}.$$

Have: $\varphi'(K_x\cdot K_y)=\varphi'(K_{x\cdot y})=K_{\varphi(x\cdot y)}=K_{\varphi(x)\circ\varphi(y)}=K_{\varphi(x)}\circ K_{\varphi(y)}=\varphi'(K_x)\circ\varphi'(K_y),$ for every $K_x,K_y\in Q/\theta$, so φ' is a homomorphism of loops. More, if $\varphi'(K_x)=\varphi'(K_y)$ then, using the fact that φ is semi-admissible with respect to θ, get: $K_{\varphi(x)}=K_{\varphi(y)} \Rightarrow \varphi(x)=\varphi(y) \Rightarrow x\theta y \Rightarrow K_x=K_y$, so φ' is an injective homomorphism of loops. Let $K_y\in Q/\theta$. So as φ is surjective, there exists an element $x\in Q$: $y=\varphi(x)$, hence $K_y=K_{\varphi(x)}=\varphi'(K_x)$, i.e. φ' is an isomorphism from $(Q/\theta,\cdot)$ to $(Q/\theta,\circ)$. □

Bibliography:

1. GRECU, I., SYRBU, P. On some isostrophy invariants of Bol loops. In: *Bull. Transylv. Univ. Braşov, Ser. C.* 2012, vol.5(54), p.145-154.
2. GREER, M., KINYON, M. Pseudoautomorphisms of Bruck loops and their generalizations. In: *Commentationes Mathematicae Universitatis Carolinae*, 2012, vol.53(3), p.383-389.
3. PFLUGFELDER, H.O. *Quasigroups and loops: Itroduction*. Sigma Series in Pure Mathematics, Heldermann. 1990. 145 p.
4. ROBINSON, D.A. Bol loops. In: *Trans. Amer. Math. Soc*., 1966, vol.123, p.341-354.
5. SYRBU, P. On middle Bol loops. In: *ROMAI J.*, 2010, vol.6(2), p.229-236.
6. БЕЛОУСОВ, В. *Основы теории квазигрупп и луп*. Москва: Наука, 1967. 223 p.
7. ГВАРАМИЯ, V. Об одном классе луп. В: *Ученые записки МГПИ*, 1971, том 378, c.23-34.

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