SOLVING TWO PERSONS GAMES WITH COMPLETE AND PERFECT INFORMATIONS

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In this article we study the non-informational two person game with complete and perfect information. The perfect information .stipulate that we can use the informational extended strategies generated by a two-directional informational flow. We propouse a new method for solving these games using Harsanyi-Selten principle.

Keywords: noncooperative game, payoff function, set of strategies,informational extended games, Bayes-Nash equilibrium.

SOLUTIONAREA JOCURILOR DE DOUA PERSOANE IN INFORMATIE COMPLETA SI PERFECTA

În acest articol se propune un algoritm nou de soluţionare a jocurilor noncooperatiste de două persoane în informaţie completă şi perfectă. Informaţia perfectă despre alegerea strategiilor permite deja utilizarea altor tipuri de strategii, strategii informaţional extinse. Pentru soluţionarea jocurilor în strategii informaţional extinse este utilizat principiul Harsanyi-Selten.

Cuvinte cheie: jocuri noncooperatiste, functii de utilitate, multime de strategii, joc informational extins, echilibru de tip Bayes-Nash.

1 Two persons game with the informational extended strategies

Let

$$\Gamma = \langle I = \{1, 2\}; X, Y; H_i : X \times Y \to R, i \in I\rangle \tag{1}$$

be the strategic form or normal form of the static noncooperative games with **complete and perfect information**, where $I = \{1, 2\}$ is the set of players, X is a set of available alternatives of the player 1, Y is a set of available alternatives of the player 2, $H_i : X \times Y \to R, i \in I$, is the payoff function of the player $i \in I$. So the players know exactly their and of the other player payoff functions and they know the sets of strategies. Players *1* and *2* know what kind of the strategy will be chosed by each other's. These conditions stipulate that we can use the informational extended strategies generated by a two-directional informational flow, denoted by $1 \overset{\inf}{\leftrightarrows} 2$, which means: at any time simultaneously player 1 knows exactly what value of the strategy will be chosed by the player 2 and player 2 knows exactly what value of the strategy which will be chosen by the player 1. In [1] the author studied these kind of games, called "the informational extensions of the games (1)". We mention that the game is static, in other words, the order of the chosen strategies is not significant. The players do not known the informational type of the other player, so the player 1 (respectively 2) does not know that the player 2 (respectively 1) knows what value of the strategies which will be chosen. In the general case [2, 3] the set of the informational extended strategies of the player 1 (respectively 2) is the set of the functions $\Theta_1 = \{\theta_1 : Y \to X\}$ (respectively $\Theta_2 = \{\theta_2 : X \to Y\}$) such that $\forall y \in Y$, $\theta_1(y) \in X$ (respectively $\forall x \in X$, $\theta_2(x) \in Y$).

To solve the informational extended game we can use the following approach.

- **Solving the informational extended game by means of the normal form.** According to [1] the payoff functions of the players will be $\mathcal{H}_i : \Theta_1 \times \Theta_2 \to R$ for all $i \in I = \{1.2\}$ and is defined as follows

$$\mathcal{H}_i(\theta_1, \theta_2) = \begin{cases} \max\limits_{(x,y) \in [gr\theta_1 \bigcap gr\theta_2]} H_i(x, y) & if\ gr\theta_1 \bigcap gr\theta_2 \neq \varnothing, \\ -\infty & if\ gr\theta_1 \bigcap gr\theta_2 = \varnothing, \end{cases}$$

where $gr\theta_1$, $gr\theta_2$ denote the graphs of the informational extended strategies $\theta_1$and θ_2. So the informational extended game $\Gamma\left(1 \overset{\inf}{\leftrightarrows} 2\right) = \left\langle I, \{\Theta_i\}_{i\in I}, \{\mathcal{H}_i\}_{i\in I}\right\rangle$ studied in [1], is the game in complete information (the players know exactly their payoff functions) and imperfect information because do not know what kind of the strategy will be chose by each other.

- **Solving the informational extended game by means of the informational non-extended game.** According to [4], the normal form game

$$\Gamma(\theta_1, \theta_2) = \left\langle I, X, Y, \left\{\widetilde{H}_i\right\}_{i\in I}\right\rangle \tag{2}$$

  where the payoff functions are defined as : $\widetilde{H}_i : X \times Y \to R$ where for all $x \in X$, $y \in Y$ we have $\widetilde{H}_i(x, y) \equiv H_i(\theta_1(y), \theta_2(x))$, will be called informational non extended game generated by the informational extended strategies θ_1 and θ_2 of the $1 \overset{\inf}{\leftrightarrows} 2$, informational extended game. The game $\Gamma(\theta_1, \theta_2)$ is played as follows: independently and simultaneously each player $i \in I$ chooses the informational non-extended strategy $x \in X$, $y \in Y$, after that the players 1 and 2 calculate the value of the informational extended strategies $\theta_1(y)$ and $\theta_2(x)$, after that each player calculates the payoff values $H_i(\theta_1(y), \theta_2(x))$, and with this the game is finished. To all strategy profiles (x, y) in the game (2) the following realization $(\theta_1(y), \theta_2(x))$ in terms of the informational extended strategies corresponds.

Suppose then that the payoff functions of the players are defined as following: for all $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$,

$$\mathcal{H}_i(\theta_1, \theta_2) = H_i(\theta_1(y), \theta_2(x)) \text{ for all } x \in X, y \in Y. \tag{3}$$

The game is played as follows: independently and simultaneously each player $i \in I = \{1, 2\}$ chooses the informational extended strategy $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$ (and players do not know what kind of the informational extended strategy will be chosed by each other's), after that the players 1 and 2 calculate the value of the payoff values $H_i(\theta_1(y), \theta_2(x))$, and with this the game is finished.

Remark 1 *The game described above will be denoted by Game* $\left(1 \overset{\inf}{\leftrightarrows} 2\right)$ *and is the game with* ***incomplete information*** *because players do not know what kind of the informational extended strategy* $\theta_1 \in \Theta_1$,*will be chosen by the player* 1 *(for example) and so the player* 1 *generates the uncertainty of the player* 2 *about the complete structure of the payoff function* $H_2(\theta_1(y), \theta_2(x))$ ***in the game with non informational extended strategies.*** *So the players do not know exactly the structure of yours payoff functions and the game is in the incomplete information. That is why for solving the game Game* $\left(1 \overset{\inf}{\leftrightarrows} 2\right)$ *the Harsanyi -Selten principle will be used.*

2 The general principle to solve the informational incomplete non-cooperative games

Here we present some basic informations about the approaches for studying and solving the informational incomplete noncooperative games.

Informally, a game of incomplete information is a game where the players do not have common knowledge of the game being played [5].This idea is tremendously important in capturing many economic situations, where a variety of features of the environment may not be commonly known. Among the aspects of the game that the players might not have common knowledge of are: payoffs; who the other players are; what moves are possible; how outcome depends on the action; what opponent knows, and what they know I know, etc. A game with incomplete information, on the other hand, tries to model situations in which some players have private information before the game begins. Following Harsanyi's 1967-68 trilogy [6, 7, 8] we can reduce the analysis of a game with incomplete information to the analysis of a game with complete (yet imperfect) information, which is fully accessible to the usual analytical tools of game theory. Harsanyi postulates that each player may be of several possible types where a type specifies all the information that the player has about the game.

The initial private information is called *the type of the player.* A type is a complete description of all relevant characteristics of a given player. Given this definition of a player's type, saying that player i knows his or her own payoff function is equivalent to saying that player i knows his or her type. Harsanyi introduces the term *Bayesian game* for the formulation of the incomplete information game by means of player type. So Harsanyi define I-game or a Bayesian game with incomplete information to be a mathematical model consisting of:

1. a set of players $I = \{1, 2\}$;
2. a set of possible actions for each player $A_i, i \in I$, $A = A_1 \times A_2$;
3. a set of possible types for each player $\Delta_i, i \in I$, $\Delta = \Delta_1 \times \Delta_2$;
4. a probability function that specifies, for each possible type of each player, a probability distribution over the other players' possible types, describing what each type of each player would believe about the other players' types $p : \Delta_1 \to \Omega(\Delta_2)$, $q : \Delta_2 \to \Omega(\Delta_1)$,where $\Omega(\Delta_2)$ (respectively $\Omega(\Delta_1)$) denotes the set of all probability distributions on a set Δ_1 (respectively Δ_2);
5. a payoff function that specifies each player's expected payoff for every possible combination of all players' actions and types $u_i : A \times \Delta \to R$.

The function p (respectively q) summarizes what player 1 (respectively player 2), given his type, believes about the types of the other players. So, $p(\delta_2|\delta_1) = \dfrac{p(\delta_2 \cap \delta_1)}{p(\delta_1)}$ (Bayes'Rule) (respectively $q(\delta_1|\delta_2) = \dfrac{q(\delta_1 \cap \delta_2)}{q(\delta_2)}$) is the conditional probability assigned to the type $\delta_2 \in \Delta_2$ (respectively $\delta_1 \in \Delta_1$) when the type of the player 1 is δ_1 (respectively of the player 2 is δ_2). Similarly, $u_i(a|\delta)$ is the payoff of player i, when the action profile is a and the type profile is δ. Here $a = (a_1, a_2) \in A$ and $\delta = (\delta_1, \delta_2) \in \Delta$. We call a Bayesian game finite if I, A_i and Δ_i are all finite, for all $i \in I$. A pure strategy for player i in a Bayesian game is a function which maps player i's type into her action set $s_i : \Delta_i \to A_i$, (strategy is a decision rule), so that $s_i(\delta_i)$ is the action choice of type δ_i of player i. Write $S_i(\Delta_i) = \{s_i : \Delta_i \to A_i | \forall \delta_i \in \Delta_i, s_i(\delta_i) \in A_i\}$ for the set of Bayesian pure strategies of the player i. Importantly, throughout in Bayesian games, the strategy spaces, the payoff functions, possible types, and the prior probability distribution are assumed to be **common knowledge**. Finally we can use the following definition of the Bayesian Normal Form of a Bayesian Game.

Definition 2 *A two persons game with incomplete information (or Bayesian game) is a game with the normal form* $\Gamma_B = \langle I = \{1, 2\}, \Delta, S, p, q, u) \rangle$ *that consists of:*

1. *a set* $\Delta = \Delta_1 \times \Delta_2$*, where* Δ_1, Δ_2 *are the finite sets of possible types for player* 1 *and respectively for player* 2;
2. *a set* $S = S_1(\Delta_1) \times S_2(\Delta_2)$*, where* $S_1(\Delta_1)$, $S_2(\Delta_2)$ *are the sets of possible strategies for player* 1 *and respectively* 2;
3. *a joint probability distribution* p, q *over types;*
4. *payoff functions* $u_1 : S \times \Delta \to R$, $u_2 : S \times \Delta \to R$ *of the player* 1*, respectively* 2*, and* $u = (u_1, u_2)$.

Mention that the Bayesian Normal Form of a Bayesian Game is the normal form representation of the game in which Nature moves first to select player types according to the common priors.

Reinhard Selten and John C. Harsanyi [9] proposed a representation of Bayesian games that enables a Bayesian game to be transformed to a strategic form game with **complete** but **imperfect** information. Each player in the original Bayesian game is now replaced with a number of type-players; in fact, a player is replaced by exactly as many type-players as the number of types in the type set of that player. We can safely assume that the type sets of the players are mutually disjoint. Let $\Delta_1 = \left\{\delta_1^1, ..., \delta_1^j, ..., \delta_1^{m_1}\right\}$and $\Delta_2 = \left\{\delta_2^1, ..., \delta_2^j, ..., \delta_2^{m_2}\right\}$. Then the set of all type-players is $J = \{j = (i, \delta_i) | \forall \delta_i \in \Delta_i, i = 1, 2\}$ and is equal to $J = \{1, ..., m_1, m_1 + 1, ..., m_1 + m_2\}$. For convenience denote by J_i the set of possible types for player i, so $J_1 = \{1, ..., m_1\}$ and $J_2 = \{1, .., m_2\}$. For all strategies $s_1(\cdot), s_2(\cdot)$ of the players 1 and 2 in the Bayesian game $\Gamma_B = \langle I = \{1, 2\}, \Delta, S, p, q, u) \rangle$ the strategies of the type-player j is defined as $r_j = \begin{cases} s_1(\delta_1^j) \in A_1 & j \in J_1, \\ s_2(\delta_2^j) \in A_2 & j \in J_2 \end{cases}$ and denote by R_j the set of pure strategies of the player j. It is clear that

$$R_j = \begin{cases} S_1(\Delta_1) & j \in J_1, \\ S_2(\Delta_2) & j \in J_2. \end{cases} \tag{4}$$

So an action profile is of the form

$$r = (r_1, ..., r_j, ..., r_{m_1+m_2}) \equiv \left(s_1(\delta_1^1), ..., s_1(\delta_1^{m_1}), s_1(\delta_2^{m_1+1}), ..., s_1(\delta_2^{m_1+m_2})\right) \in R = \prod_{j \in J} R_j.$$

The payoff function U_j is the conditionally expected utility to player $i \in I$ in the Bayesian game given that δ_i^j is his actual type. Denote by $S_i\left(\delta_i^j\right) = \left\{s_i \in S_i(\Delta_i) : |s_i(\delta_i^j) \in A_i\right\}$ that are the sets of all allowable alternatives of the player i, that is of type δ_i^j (a type-player j). So for all type-player $j \in J_1$ payoff function $U_j : S_1\left(\delta_1^j\right) \times S_2(\Delta_2)$ is defined as following

$$U_j\left(r_j, \{r_k\}_{k \in J_2}\right) = U_j\left(s_1(\delta_1^j), \left\{s_2(\delta_2^k)\right\}_{k \in J_2}\right) = \sum_{k \in J_2} p(\delta_2^k | \delta_1^j) u_1\left(s_1(\delta_1^j), s_2(\delta_2^k), \delta_1^j, \delta_2^k\right). \tag{5}$$

Similar for all type-player $k \in J_2$ payoff function $U_k : S_1(\Delta_1) \times S_2\left(\delta_2^k\right)$ is defined as following

$$U_k\left(\{r_j\}_{j \in J_1}, r_k\right) = U_k\left(\left\{s_1(\delta_1^j)\right\}_{j \in J_1}, s_2(\delta_2^k)\right) = \sum_{j \in J_1} q(\delta_1^j | \delta_2^k) u_2\left(s_1(\delta_1^j), s_2(\delta_2^k), \delta_1^j, \delta_2^k\right). \tag{6}$$

Suppose also, that Nature, using the probability distributions p, q, randomly chooses which of these types actually will play the game and each type of every player must choose her strategy before Nature's random choice.

So, according to Harsanyi's 1967-68 trilogy [6, 7, 8], we can state the following definition.

Definition 3 *Given a Bayesian game $\Gamma_B = \langle I = \{1, 2\}, \Delta, S, p, q, u) \rangle$, then an equivalent strategic form game is $\Gamma_B^* = \left\langle J, \{R_j\}_{j \in J}, \{U_j\}_{j \in J} \right\rangle$ whose type-player set is J, player j's strategy set is defined by (4) and payoff function are defined by (5-6).*

We then say that Γ_B^* is the strategic form game associated with the incomplete information game Γ_B. So according to fundamental observation by Harsanyi, games of incomplete information $\Gamma_B = \langle I = \{1, 2\}, \Delta, S, p, q, u) \rangle$ (Bayesian game) can be thought of as games of complete, but imperfect information $\Gamma_B^* = \left\langle J, \{R_j\}_{j \in J}, \{U_j\}_{j \in Jj} \right\rangle$ (i.e. the player has only partial information about the actions taken previously by another player, some of the previous moves by other players are not observed, when a player is called upon to move where nature makes the first move (selecting δ_1, δ_2), but not everyone observes nature's move (i.e. player 1 learns δ_1 but not δ_2).

Remark 4 *The notion of "type-player" means the following: type-player j is the player 1 (or the player 2) that with probability $p(\delta_2|\delta_1)$ (respectively $q(\delta_1|\delta_2)$) has a complete information about the normal form of the game.*

According to Harsanyi, a Bayesian equilibrium specifies a (possibly randomized) action for each possible type of each player, such that each type's specified action maximizes his conditional expected payoff given his type, given his beliefs about the other players' types, and given the type-contingent behavior of all other players according to this equilibrium. A Bayesian Nash Equilibrium of the game Γ_B is simply a Nash Equilibrium of the game Γ_B^* where Nature moves first, chooses $\delta = (\delta_1, \delta_2) \in \Delta$ from a distribution with probabilities $p(\delta_2|\delta_1)$, $q(\delta_1|\delta_2)$ and reveals δ_i to player $i = 1, 2$.

So for solving the incomplete information game of type $\Gamma_B = \langle I = \{1,2\}, \Delta, S, p, q, u \rangle\rangle$, it is sufficient to determine the Nash equilibrium profiles in the complete informational game type $\Gamma_B^* = \left\langle J, \{R_j\}_{j\in J}, \{U_j\}_{j\in J} \right\rangle$. In general case, strategy profile $r^* = \left(\left\{ r_j^* \right\}_{j\in J_1}, \{r_k^*\}_{k\in J_2} \right) \equiv$ $\equiv \left(\left\{ s_1^*(\delta_1^j) \right\}_{j\in J_1}, \{s_2^*(\delta_2^k)\}_{k\in J_2} \right)$ is Nash equilibrium profile in the game $\Gamma_B^* = \left\langle J, \{R_j\}_{j\in J}, \{U_j\}_{j\in J} \right\rangle$ if for all $j \in J_1$, $k \in J_2$, $\delta_1^j \in \Delta_1 . \delta_2^k \in \Delta_2$ the following conditions are fulfilled

$$\begin{cases} U_j\left(r_j^*, \{r_k^*\}_{k\in J_2}\right) = U_j\left(s_1^*(\delta_1^j), \{s_2^*(\delta_2^k)\}_{k\in J_2}\right) \geqslant U_j\left(s_1(\delta_1^j), \{s_2^*(\delta_2^k)\}_{k\in J_2}\right) \forall s_1 \in S_1\left(\delta_1^j\right), \\ U_k\left(\left\{r_j^*\right\}_{j\in J_1}, r_k^*\right) = U_k\left(\left\{s_1^*(\delta_1^j)\right\}_{j\in J_1}, s_2^*(\delta_2^k)\right) \geqslant U_k\left(\left\{s_1^*(\delta_1^j)\right\}_{j\in J_1}, s_2(\delta_2^k)\right) \forall s_2 \in S_2\left(\delta_2^k\right). \end{cases} \tag{7}$$

According to (5)-(6) from (7) we have that the strategy profile $\left(\left\{ s_1^*(\delta_1^j) \right\}_{j\in J_1}, \{s_2^*(\delta_2^k)\}_{k\in J_2} \right)$ is Nash equilibrium in the game $\Gamma_B^* = \left\langle J, \{R_j\}_{j\in J}, \{U_j\}_{j\in J} \right\rangle$ if for all $j \in J_1$, $k \in J_2$, $\delta_1^j \in \Delta_1 . \delta_2^k \in \Delta_2$ the following conditions are fulfilled

$$\begin{cases} \sum\limits_{k\in J_2} p(\delta_2^k|\delta_1^j) u_1\left(s_1^*(\delta_1^j), s_2^*(\delta_2^k), \delta_1^j, \delta_2^k\right) \geqslant \sum\limits_{k\in J_2} p(\delta_2^k|\delta_1^j) u_1\left(s_1(\delta_1^j), s_2^*(\delta_2^k), \delta_1^j, \delta_2^k\right) \forall s_1 \in S_1\left(\delta_1^j\right), \\ \sum\limits_{j\in J_1} q(\delta_1^j|\delta_2^k) u_2\left(s_1^*(\delta_1^j), s_2^*(\delta_2^k), \delta_1^j, \delta_2^k\right) \geqslant \sum\limits_{j\in J_1} q(\delta_1^j|\delta_2^k) u_2\left(s_1^*(\delta_1^j), s_2(\delta_2^k), \delta_1^j, \delta_2^k\right) \forall s_2 \in S_2\left(\delta_2^k\right). \end{cases} \tag{8}$$

Finally, because the affirmation "$\forall s_1 \in S_1\left(\delta_1^j\right)$" (respectively "$s_2 \in S_2\left(\delta_2^k\right)$") is equivalent to affirmation "$\forall a_1 \in A_1$" (respectively " "$\forall a_2 \in A_2$") from (8) we obtain that the strategy profile $\left(\left\{ s_1^*(\delta_1^j) \right\}_{j\in J_1}, \{s_2^*(\delta_2^k)\}_{k\in J_2} \right)$ is Nash equilibrium in the game $\Gamma^* = \left\langle J, \{R_j\}_{j\in J}, \{U_j\}_{j\in J} \right\rangle$ if for all $j \in J_1$, $k \in J_2$, $\delta_1^j \in \Delta_1 . \delta_2^j \in \Delta_2$ the following conditions are fulfilled

$$\begin{cases} \sum\limits_{k\in J_2} p(\delta_2^k|\delta_1^j) u_1\left(s_1^*(\delta_1^j), s_2^*(\delta_2^k), \delta_1^j, \delta_2^k\right) \geqslant \sum\limits_{k\in J_2} p(\delta_2^k|\delta_1^j) u_1\left(a_1, s_2^*(\delta_2^k), \delta_1^j, \delta_2^k\right) \forall a_1 \in A_1, \\ \sum\limits_{j\in J_1} q(\delta_1^j|\delta_2^k) u_2\left(s_1^*(\delta_1^j), s_2^*(\delta_2^k), \delta_1^j, \delta_2^k\right) \geqslant \sum\limits_{j\in J_1} q(\delta_1^j|\delta_2^k) u_2\left(s_1^*(\delta_1^j), a_2, \delta_1^j, \delta_2^k\right) \forall a_2 \in A_2. \end{cases} \tag{9}$$

3 Converting the two persons game with informational extended strategies to Bayesian game.

Let $Game\left(1 \overset{\inf}{\leftrightarrows} 2\right)$ is the two persons informational extended game with the following sets of the informational extended strategies $\Theta_1 = \left\{ \theta_1^j : Y \to X | \forall y \in Y, \theta_1^j(y) \in X, j = \overline{1, m_1} \right\}$ of the player 1 and $\Theta_2 = \left\{\theta_2^k : X \to Y | \forall x \in X, \theta_2^k(x) \in Y, k = \overline{1, m_2}\right\}$ of the player 2. Denote also by $\widetilde{X}_j = \left\{\widetilde{x}_j \in X : \widetilde{x}_j = \theta_1^j(y), \forall y \in Y\right\}$ and $\widetilde{Y}_k = \left\{\widetilde{y}_k \in Y : \widetilde{y}_k = \theta_2^k(x), \forall x \in X\right\}$ the set of of all range of the informational extended strategy θ_1^j of the player 1 and θ_2^k of the player 2. The sets $\widetilde{X}_j$ and $\widetilde{Y}_k$ are the sets of informational non extended strategies generated by the informational extended strategies of the player 1 and 2 respectively. According to the above mentioned, we can reduce the analysis of a game with incomplete information to the analysis of a game with complete (but imperfect) information, which is fully accessible to the usual analytical tools of game theory. So to solve the game $Game\left(1 \overset{\inf}{\leftrightarrows} 2\right)$ we must do the following step-intervals:

1. Construct the Bayesian game $\Gamma_B = \left\langle I = \{1, 2\}, S_1(\Delta_1), S_2(\Delta_2), \Delta_1, \Delta_2, p, q, \widetilde{H}_1, \widetilde{H}_1 \right\rangle$ that corresponds (is associated) to the game $Game\left(1 \overset{\inf}{\leftrightarrows} 2\right)$.
2. For game Γ_B from step 1 construct the Selten-Harsanyi game $\Gamma_B^* = \left\langle J, \{R_j\}_{j\in J}, \{U_j\}_{j\in Jj} \right\rangle$ with complete and imperfect information on the sets of the non-informational extended strategies.
3. Determine Nash equilibrium profiles in the game Γ_B^* that is the Bayes-Nash equilibrium in the game Γ_B.
4. As the solution of the game $Game\,(1 \Leftrightarrow 2)$ we will consider the non informational extended strategy profile (x^*, y^*) which is generated by the Nash strategy profile in the game Γ_B^*.

According to definition 2 we construct for game $Game\left(1 \overset{\inf}{\leftrightarrows} 2\right)$ the corresponding Bayesian game. The normal form must consisting of the following.

- A set of possible actions for the player 1 is X, a for player 2 is Y ;
- Because players do not know what kind of the informational extended strategy will be chosen by the other player, then the uncertainty of the player 1 about the own payoff function structure is generated by the player 2 selected informational extended strategy and respectively, the uncertainty of the player 2 about the own payoff function structure is generated by the player 1 selected informational extended strategy. The set of types for player *1* (player *2*) is $\Delta_1 = \{\delta_1^j, j \in J_1\}(\Delta_2 = \{\delta_2^k, k \in J_2\})$. In other words, the player 1(player 2) is of the type δ_1^j (of the type δ_2^k) if he generates to the player 2 (to the player 1) payoff structure uncertainty, selecting the $\theta_1^j \in \Theta_1$ ($\theta_2^k \in \Theta_2$) informational extended strategy.
- The probability function $p : \Delta_1 \to \Omega(\Delta_2)$ of the player 1, respectively $q : \Delta_2 \to \Omega(\Delta_1)$ of the player 2, means the following: if the player 1(player 2) chooses the informational extended strategy θ_1^j (strategy θ_2^k), then he believes that the player 2 (player 1) with the $p(\delta_2^k/\delta_1^j) = \dfrac{p(\delta_2^k \cap \delta_1^j)}{p(\delta_1^j)}$ (respectively $q(\delta_1^j/\delta_2^k) = \dfrac{q(\delta_1^j \cap \delta_2^k)}{q(\delta_2)}$) chooses the informational extended strategy θ_2^k (strategy θ_1^j).
- The set of the strategies are the set of all range of the informational extended strategies of the players

$$\mathcal{S}_1(\Delta_1) = \left\{\widetilde{x}_j \in X : \widetilde{x}_j = \theta_1^j(y), \forall y \in Y, \forall j \in J_1\right\} \equiv \left\{\widetilde{X}_j, j = 1, ..., m_1\right\}, \tag{10}$$

$$\mathcal{S}_2(\Delta_2) = \left\{\widetilde{y}_k \in Y : \widetilde{y}_k = \theta_2^k(x), \forall x \in X, , \forall k \in J_2\right\} \equiv \left\{\widetilde{Y}_k, k = 1, ..., m_2\right\}. \tag{11}$$

So if the player 1,for example, is of the type j, i.e. he chooses the informational extended strategy θ_1^j, then the set of the informational non extended strategies, generated by θ_1^j is $\widetilde{X}_j$.

- According to the sets of strategies (10)-(11), the payoff functions of the player is defined as following $\widetilde{H}_1 : S_1(\Delta_1) \times S_2(\Delta_2) \times \Delta_1 \times \Delta_2 \to R$, $\widetilde{H}_2 : S_1(\Delta_1) \times S_2(\Delta_2) \times \Delta_1 \times \Delta_2 \to R$. More exact, for all fixed $s_1 \in S_1(\Delta_1)$ and $s_2 \in S_2(\Delta_2)$,

$$\widetilde{H}_1(s_1(\cdot), s_2(\cdot), \delta_1^j, \delta_2^k) = H_1\left(\widetilde{x}_j, \widetilde{y}_k, \delta_1^j, \delta_2^k\right), \quad (12)$$

$$\widetilde{H}_2(s_1(\cdot), s_2(\cdot), \delta_1^j, \delta_2^k) = H_2\left(\widetilde{x}_j, \widetilde{y}_k, \delta_1^j, \delta_2^k\right) \quad (13)$$

for all $\widetilde{x}_j \in \widetilde{X}_j$, $\widetilde{y}_k \in \widetilde{Y}_k$, $j = \overline{1, m_1}$, $k = \overline{1, m_2}$.

So for the game *Game* $(1 \Leftrightarrow 2)$ we construct the following normal form of the associated Bayesian game in the non extended strategies

$$\Gamma_B = \left\langle I = \{1,2\}, \mathcal{S}_1(\Delta_1), \mathcal{S}_2(\Delta_2), \Delta_1, \Delta_2, p, q, \widetilde{H}_1, \widetilde{H}_2 \right\rangle \quad (14)$$

were the sets of strategies is defined by (10)-(11) and payoff functions is defined by (12)-(13). Denote by $BE\left[\Gamma_B\right]$ the set of all Bayes-Nash strategies profile of the game Γ_B. Really, solving the game Γ_B is difficult, because it is transformed in the two level dynamic game. On the first level player Nature chooses the informational extended strategies of the players, for example $\left(\theta_1^j, \theta_2^k\right)$, and on the second level each player choose the informational non extended strategies from the set $\widetilde{X}_j$ (player 1) and from the set $\widetilde{Y}_k$ (player 2). The games $\Gamma\left(\theta_1^j, \theta_2^k\right)$ are the subgames in the dynamic game defined above. In these article we do not investigate such method to solve informational extended game.

According to the definition 3 we construct the game Γ_B^*, which will be solved. Denote by $J = \left\{j = (i, \delta_i^j), i = 1, 2, j = 1, ..., m_1 + m_2\right\}$ the set of type-players that is equal to the sets of all informational extended strategies of the players $J = J_1 \cup J_2$. The strategy of the type-player j is $r_j = \begin{cases} \widetilde{x}_j \in \widetilde{X}_j & j \in J_1, \\ \widetilde{y}_j \in \widetilde{Y}_j & j \in J_2 \end{cases}$ and means the following: if player is of type $j = (i, \delta_i^j)$ (i.e. player $i, i = 1, 2$, chooses the informational extended strategy θ_i^j), then the strategy will be equal to value of the informational extended strategy $x^j = \theta_i^j(y)$ for a fixed value of the non extended strategy $y \in Y$. The sets of pure strategies of the players will be $R_j = \begin{cases} \widetilde{X}_j & j \in J_1, \\ \widetilde{Y}_j & j \in J_2. \end{cases}$, and $R = \prod_{j=1}^{m_1+m_2} R_j$. For all type-player $j = (1, \delta_1^j)$, $j \in J_1$, payoff function $U_j : \widetilde{X}_j \times \left(\prod_{k\in J_2} \widetilde{Y}_k\right)$ is defined as following

$$U_j\left(r_j, \{r_k\}_{k\in J_2}\right) = U_j\left(\widetilde{x}_j, \{\widetilde{y}_k\}_{k\in J_2}\right) = \sum_{k\in J_2} p\left(\delta_2^k | \delta_1^j\right) H_1\left(\widetilde{x}_j, \widetilde{y}_k\right) \ \forall \widetilde{x}_j \in \widetilde{X}_j, \widetilde{y}_k \in \widetilde{Y}_k. \quad (15)$$

In similar mode for all type-player $j = (2, \delta_2^j)$, $j \in J_2$ payoff function $U_j : \left(\prod_{k\in J_1} \widetilde{X}_k\right) \times \widetilde{Y}_j$ is defined as following

$$U_j\left(\{r_k\}_{k\in J_1}, r_j\right) = U_j\left(\{\widetilde{x}_k\}_{k\in J_1}, \widetilde{y}_j\right) = \sum_{k\in J_1} q\left(\delta_1^k | \delta_2^j\right) H_2\left(\widetilde{x}_k, \widetilde{y}_j\right) \ \forall \widetilde{x}_k \in \widetilde{X}_k, \widetilde{y}_j \in \widetilde{Y}_j. \quad (16)$$

We make the following

Remark 5 *Utility functions of type (15)-(16) have the following meaning. If, for example, the player 1, had chosen information extended strategy* θ_1^j, *which also means he has a type-player* $j = (1, \delta_1^j)$, *and with the probability* $p(\delta_2^k|\delta_1^j)$ *he assumes that the player* 2 *will choose the information extended strategy*

θ_2^k, *i.e. as we have the type player* $k = (2, \delta_2^k)$, *for all* $k \in J_2$, *then for all information not extended strategy* $x \in X$ *and* $y \in Y$, *average value of the payoff will be equal to*

$$\sum_{k \in J_2} p(\delta_2^k|\delta_1^j) H_1 (\widetilde{x}_j, \widetilde{y}_k) \equiv \sum_{k \in J_2} p(\delta_2^k|\delta_1^j) H_1 \left(\theta_1^j(y), \theta_2^k(x) \right).$$

Since $p(\delta_1^l|\delta_1^j) = 0$ *for all* $l \in J_1$, $l \neq j$ *in the right side of relation (15) thre are no terms with* $l \in J_1$, $l \neq j$.

So, we obtain the following game with complete and imperfect information on the set of non informational extended strategies generated by the informational extended strategies. This game will be called Selten-Harsanyi type game and will have the following normal form

$$\Gamma_B^* = \left\langle J, \{R_j\}_{j \in J}, \{U_j\}_{j \in J} \right\rangle. \tag{17}$$

The game (17) is played as follows: for all fixed probabilities $p(\delta_2^k|\delta_1^j)$ and $q\left(\delta_1^k|\delta_2^j\right)$, independently and simultaneously each type-player $j = (i, \delta_i^j)$ chooses the strategy $r_j \in R_j$, after that each player calculates the payoff using the functions (15) or (16) and whereupon the game is finished. In other words, because strategies r_j are defined by the informational non extended strategies from sets X and Y, for all $y \in Y$ (respectively for all $x \in X$) type-player $j = (1, \delta_1^j)$ (respectively type-player $k = (2, \delta_2^k)$) chooses the strategy $\widetilde{x}_j = \theta_1^j(y)$ (respectively $\widetilde{y}_k = \theta_2^k(x)$), calculates the payoff values using the functions (15) (respectively (16)). and with this the game is finished

Using relations (9) we introduce the following definition.

Definition 6 *Strategy profile* $r^* = \left(r_1^*, ..., r_j^*, ...r_{|J|}^*\right)$ *is the Nash equilibrium in the game* $\Gamma_B^* = \left\langle J, \{R_j\}_{j \in J}, \{U_j\}_{j \in J} \right\rangle$ *if an only if the following conditions are fulfilled:*

$$\begin{cases} U_j \left(r_j^*, \{r_k^*\}_{k \in J_2}\right) \geqslant U_j \left(r_j, \{r_k^*\}_{k \in J_2}\right) \text{ for all } j \in J_1, \\ U_j \left(\{r_k^*\}_{k \in J_1}, r_j^*\right) \geqslant U_j \left(\{r_k^*\}_{k \in J_1}, r_j\right) \text{ for all } j \in J_2. \end{cases} \tag{18}$$

Denote by $NE\left[\Gamma_B^*\right]$ the set of all Nash equilibrium strategies profile in the game Γ_B^*. Using relations (15)-(16) we get for all fixed probabilities $p(\cdot), q(\cdot)$ that strategy profile $\left(\left\{\widetilde{x}_j^*\right\}_{l \in J_1}, \{\widetilde{y}_k^*\}_{k \in J_2}\right)$ is Nash equilibrium for the game $\Gamma_B^* = \left\langle J, \{R_j\}_{j \in J}, \{U_j\}_{j \in J} \right\rangle$ if and only if the $|J_1| + |J_2|$ conditions are fulfilled:

$$\begin{cases} \sum\limits_{k \in J_2} p(\delta_2^k|\delta_1^j) H_1 \left(\widetilde{x}_j^*, \widetilde{y}_k^*\right) \geqslant \sum\limits_{k \in J_2} p\left(\delta_2^k|\delta_1^j\right) H_1 (\widetilde{x}_j, \widetilde{y}_k^*) \;\; \forall \widetilde{x}_j \in \widetilde{X}_j, \; j \in J_1, \\ \sum\limits_{j \in J_1} q(\delta_1^j|\delta_2^k) H_2 \left(\widetilde{x}_j^*, \widetilde{y}_k^*\right) \geqslant \sum\limits_{j \in J_1} q(\delta_1^j|\delta_2^k) H_2 \left(\widetilde{x}_j^*, \widetilde{y}_k\right) \;\; \forall \widetilde{y}_k \in \widetilde{Y}_k, \; k \in J_2. \end{cases} \tag{19}$$

The relation between the Nash equilibrium in the Harsanyi game Γ_B^* and the equilibrium at the Bayesian game Γ_B was given by Harsanyi [6-8]

Theorem 7 (Hansanyi) *The set of Nash equilibria of the game* Γ_B^* *is identical to the set of Bayesian equilibria of the game* Γ_B.

Let strategy profile $\left(\left\{\widetilde{x}_j^*\right\}_{j \in J_1}, \{\widetilde{y}_k^*\}_{k \in J_2}\right) \in NE\left[\Gamma_B^*\right]$, then using Remark 5 we can introduce the following definition.

Definition 8 *For all fixed probabilities $p(\cdot), q(\cdot)$ strategy profile $(x^*, y^*) \equiv (x^*(p), y^*(q))$ $x^* \in X$, $y^* \in Y$, for which the following conditions*

$$\begin{cases} \widetilde{x}_j^* = \theta_1^j(y^*) \ \forall j \in J_1 \\ \widetilde{y}_k^* = \theta_2^k(x^*) \ \forall k \in J_2 \end{cases} \tag{20}$$

are fulfilled, is called the Bayes-Nash equilibrium profile in non informational extended strategies of the game Γ from (1), generated by the $\left(\left\{\widetilde{x}_j^*\right\}_{j\in J_1}, \{\widetilde{y}_k^*\}_{k\in J_2}\right) \in NE\left[\Gamma_B^*\right]$

Denote by $BN\left[G\left(1 \leftrightarrows 2\right)\right]$ the set of all Bayes-Nash equilibrium profiles in the game $G\left(1 \leftrightarrows 2\right)$ .So, such as a solutions of the informational extended games $Game\left(1 \overset{\inf}{\leftrightarrows} 2\right)$ we consider the informational non extended Bayes-Nash equilibrium profiles $(x^*, y^*) \equiv (x^*(p), y^*(q))$ for which the relations (20) from Definition 2 are fulfilled for all fixed probabilities $p(\cdot)$and $q(\cdot)$of the believes about the choice of the informational extended strategy of the other player.

The following example illustrate the above context.

Example 9 *Consider the two persons game in the complete and perfect information, for which $X = [0,1]$, $Y = [0,1]$, are the sets of strategies and $h_1(x,y) = \frac{3}{2}xy - x^2$, $h_2(x,y) = \frac{3}{2}xy - y^2$ the payoff functions of the player. Solve this game using the described above Harsanyi's principle.*

Solution. In the capacity of the informational extended strategies we will use the functions $\theta_1 : Y \to X$, where for $\forall y \in Y$, $\theta_1(y) = \arg\max_{x\in X} h_1(x,y)$, respectively $\theta_2 : X \to Y$ where $\forall x \in X$, $\theta_2(x) = \arg\max_{y\in Y} h_2(x,y)$. Using the necessary condition of optimality, we obtain that $\theta_1(y) = \frac{3}{4}y$ $\forall y \in [0,1]$ and $\theta_2(x) = \frac{3}{4}x$ $\forall x \in [0,1]$. So, we can consider the following sets of the informational extended strategies

$$\Theta_1 = \left\{\theta_1^j : Y \to X | \forall y \in Y, \theta_1^j(y) \in X, j = \overline{1,2}\right\} \Theta_2 = \left\{\theta_2^k : X \to Y | \forall x \in X, \theta_2^k(x) \in Y, k = \overline{1,2}\right\}.$$

If players will use these strategies then the payoff functions will be $H_1(x,y) = \frac{3}{2}\theta_1(y)\theta_2(x) - (\theta_1(y))^2 = \frac{3}{2}\left(\frac{3}{4}y\right)\underbrace{\left(\frac{3}{4}x\right)} - \left(\frac{3}{4}y\right)^2 = \frac{27}{32}xy - \frac{9}{16}y^2$ and $H_2(x,y) = \frac{3}{2}\theta_1(y)\theta_2(x) - (\theta_2(x))^2 = \frac{3}{2}\underbrace{\left(\frac{3}{4}y\right)}\left(\frac{3}{4}x\right) - \left(\frac{3}{4}x\right)^2 = \frac{27}{32}xy - \frac{9}{16}x^2$. But because the player 1, for example, will not know that the player 2 as information extended strategy choose exactly $\theta_2(x) = \frac{3}{4}x$, then he (the player 1) will not know its payoff function, i.e. not knowledge "component $\theta_2(x) = \frac{3}{4}x$" of the payoff function, i.e. we can write $H_1(x,y) = \frac{3}{2}\left(\frac{3}{4}y\right)?\underbrace{\left(\frac{3}{4}x\right)}? - \left(\frac{3}{4}y\right)^2$. So the informational extended strategies generate uncertainty of the payoff functions, that we already have a incomplete information game.

Construct the Bayesian game Γ_B associated to the initial informational extended game.

a) The actions sets are $X = [0,1]$ and $Y = [0,1]$.

b) The set of the type for player 1 is $\Delta_1 = \{\delta_1^1, \delta_1^2\}$ and for player 2 is $\Delta_2 = \{\delta_2^1, \delta_2^2\}$ that means the following: player *1* (respectively *2*) is of δ_1^1 type (respectively δ_2^1) if he choose the informational extended strategy $\theta_1^1(y) = \frac{3}{4}y$ (respectively $\theta_2^1(x) = \frac{3}{4}x$) and of the δ_1^2 type (respectively δ_2^2) if he choose the informational extended strategy $\theta_1^2(y) = y^2$ (respectively $\theta_2^2(x) = x^2$). Type-players will be denoted by $J_1 = \{1,2\}$, $J_2 = \{1,2\}$.

c) Suppose that the types are independently distributed and the probability of the type δ_1^1 is $0 \le p \le 1$ and the probability of the type δ_2^1 is $0 \le q \le 1$. So if the player 2 is of the type δ_2^k, then he supposes with the probability $p\left(\delta_1^j/\delta_2^k\right) = p(\delta_1^j) = \begin{cases} p \text{ for } j=1 \\ 1-p \text{ for } j=2 \end{cases}$ that the player 1 is of the type δ_1^j and, respectively, if the player 1 is of the type δ_1^j, then he suppose with the probability $p(\delta_2^k/\delta_1^j) = p(\delta_2^k) = \begin{cases} q \text{ for } k=1 \\ 1-q \text{ for } k=2 \end{cases}$ that the player 2 is of the type δ_2^k.

d) According to (10)-(11), the strategies sets of the players are defined as following. If the player 1 is of type δ_1^1, then his strategy will be determined by the function $\theta_1^1(y) = \frac{3}{4}y$, and so we have the following set of strategies $S_1(\delta_1^1) \equiv \widetilde{X}_1 = \left\{\widetilde{x}_1 \in [0,1] : \widetilde{x}_1 = \frac{3}{4}y, \forall y \in [0,1]\right\} = \left[0, \frac{3}{4}\right] \subseteq X$. Similarly if player 1 is of type δ_1^2 then his strategy will be determined by the function $\theta_1^2(y) = y^2$, and thus obtain the following set of strategies $S_1(\delta_1^2) \equiv \widetilde{X}_2 = \left\{\widetilde{x}_2 \in [0,1] : \widetilde{x}_2 = y^2, \forall y \in [0,1]\right\} = [0,1]$. If the player 2 is of type δ_2^1, then his strategy will be determined by function $\theta_2^1(x) = \frac{3}{4}x$,and thus $S_2(\delta_2^1) \equiv \widetilde{Y}_1 = \left\{\widetilde{y}_1 \in [0,1] : \widetilde{y}_1 = \frac{3}{4}x, \forall x \in [0,1]\right\} = \left[0, \frac{3}{4}\right] \subseteq Y$. Similarly, if player 2 is of type δ_2^2, then his strategy will be determined by the function $\theta_2^2(x) = x^2$, and thus we obtain the following set of strategies $S_2(\delta_2^2) \equiv \widetilde{Y}_2 = \left\{\widetilde{y}_2 \in [0,1] : \widetilde{y}_2 = x^2, \forall x \in [0,1]\right\} = [0,1]$. Finally,

$$\mathcal{S}_1(\Delta_1) = \begin{cases} S_1(\delta_1^j) \equiv \widetilde{X}_1 & j=1 \\ S_1(\delta_1^j) \equiv \widetilde{X}_2 & j=2 \end{cases}, \mathcal{S}_2(\Delta_2) = \begin{cases} S_2(\delta_2^k) \equiv \widetilde{Y}_1 & k=1 \\ S_2(\delta_2^k) \equiv \widetilde{Y}_2 & k=2 \end{cases}. \tag{21}$$

e) According to (12)-(13), the payoff functions of the player 1 is $\widetilde{H}_1\left(\widetilde{x}_j, \widetilde{y}_k, \delta_1^j, \delta_1^k\right) \equiv H_1\left(\widetilde{x}_j, \widetilde{y}_k\right)$ and $\widetilde{H}_2\left(\widetilde{x}_j, \widetilde{y}_k, \delta_1^j, \delta_1^k\right) \equiv H_2\left(\theta_1^j(y), \theta_2^k(x)\right)$ of the player 2. Finally

$$\widetilde{H}_1\left(\widetilde{x}_j, \widetilde{y}_k, \delta_1^j, \delta_1^k\right) = \begin{cases} \frac{3}{2}\widetilde{x}_1\widetilde{y}_1 - (\widetilde{x}_1)^2, & j=1, k=1, \\ \frac{3}{2}\widetilde{x}_2\widetilde{y}_1 - (\widetilde{x}_2)^2, & j=2, k=1, \\ \frac{3}{2}\widetilde{x}_1\widetilde{y}_2 - (\widetilde{x}_1)^2, & j=1, k=2, \\ \frac{3}{2}\widetilde{x}_2\widetilde{y}_2 - (\widetilde{x}_2)^2, & j=2, k=2, \end{cases} \tag{22}$$

$$\widetilde{H}_2\left(\widetilde{x}_j, \widetilde{y}_k, \delta_1^j, \delta_1^k\right) = \begin{cases} \frac{3}{2}\widetilde{x}_1\widetilde{y}_1 - (\widetilde{y}_1)^2, & j=1, k=1, \\ \frac{3}{2}\widetilde{x}_2\widetilde{y}_1 - (\widetilde{y}_2)^2, & j=2, k=1, \\ \frac{3}{2}\widetilde{x}_1\widetilde{y}_2 - (\widetilde{y}_1)^2, & j=1, k=2, \\ \frac{3}{2}\widetilde{x}_2\widetilde{y}_2 - (\widetilde{y}_2)^2, & j=2, k=2. \end{cases} \tag{23}$$

Thus we obtained the Bayesian game $\Gamma_B = \left\langle I = \{1,2\}, \mathcal{S}_1(\Delta_1), \mathcal{S}_2(\Delta_2), \Delta_1, \Delta_2, p, q, \widetilde{H}_1, \widetilde{H}_2 \right\rangle$, where the sets of the strategies is determined by the relation 21) and the payoff functions by the relations (22)-(23).

Now we can construct the game Γ_B^* in complete and imperfect informations, associated to the Bayesian game recently constructed.

aa) The set of the type-players is $J = J_1 \cup J_2$, where $J_1 = \left\{j = (1, \theta_1^j) | j = 1,2\right\} = \{1,2\}$and $J_2 = \left\{k = (2, \theta_2^k) | k = 1,2\right\} = \{3,4\}$.Finally, $J = \{1,2,3,4\}$.

bb) The set of the strategy of the type-player $j \in J$ is $R_j = \begin{cases} \widetilde{X}_j & j \in J_1, \\ \widetilde{Y}_j & j \in J_2. \end{cases}$ and the strategy of the type-player $j \in J$ is $r_j = \begin{cases} \widetilde{x}_j \in \widetilde{X}_j & j \in J_1, \\ \widetilde{y} \in \widetilde{Y}_j & j \in J_2 \end{cases}$. So, for type player $j = 1$ the strategy set is $R_1 = \widetilde{X}_1 = \left[0, \frac{3}{4}\right]$, for type-player $j = 2$ the strategy set is $R_2 = \widetilde{X}_2 = [0,1]$, for type-player $j = 3$ (or $k = 1$) $R_3 = \widetilde{Y}_1 = \left[0, \frac{3}{4}\right]$ and for type-player $j = 4$, the strategy set is $R_4 = \widetilde{Y}_2 = [0,1]$. According to these, the strategy profile in the game Γ_B^* is $r = (r_1, r_2, r_3, r_4) = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2)$, where $\widetilde{x}_1 \in \left[0, \frac{3}{4}\right]$, $\widetilde{x}_2 \in [0,1]$, $\widetilde{y}_1 \in \left[0, \frac{3}{4}\right]$ and $\widetilde{y}_2 \in [0,1]$.

cc) According to (15)- (16), payoff functions of the type-players are defined as following

$$U_1(\widetilde{x}_1, \widetilde{y}_1, \widetilde{y}_2, q) = qH_1(\widetilde{x}_1, \widetilde{y}_1) + (1-q)H_1(\widetilde{x}_1, \widetilde{y}_2) = = -(\widetilde{x}_1)^2 + \frac{3}{2}\widetilde{x}_1(q\widetilde{y}_1 + (1-q)\widetilde{y}_2), \tag{24}$$

$$U_2(\widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2, q) = qH_1(\widetilde{x}_2, \widetilde{y}_1) + (1-q)H_1(\widetilde{x}_2, \widetilde{y}_2) = = -(\widetilde{x}_2)^2 + \frac{3}{2}\widetilde{x}_2(q\widetilde{y}_1 + (1-q)\widetilde{y}_2), \tag{25}$$

$$U_3(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, p) = pH_2(\widetilde{x}_1, \widetilde{y}_1) + (1-p)H_2(\widetilde{x}_2, \widetilde{y}_1) = -(\widetilde{y}_1)^2 + \frac{3}{2}\widetilde{y}_1(p\widetilde{x}_1 + (1-p)\widetilde{x}_2), \tag{26}$$

$$U_4(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_2, p) = pH_2(\widetilde{x}_1, \widetilde{y}_2) + (1-p)H_2(\widetilde{x}_2, \widetilde{y}_2) = -(\widetilde{y}_2)^2 + \frac{3}{2}\widetilde{y}_2(p\widetilde{x}_1 + (1-p)\widetilde{x}_2). \tag{27}$$

These features mean the following: for example, if the player 1 chooses the information extended strategy $\theta_1^1(y)$ (thus we have the type-player $j = 1$) and in the assumption that the player 2 with probability q will choose the information extended strategy $\theta_2^1(x)$ and with probability $(1-q)$ the strategy $\theta_2^2(x)$, then $\forall x \in X, y \in Y$ the average payoff (in terms of average uncertainty about the payoff) of type-player 1 will be determined by the relation (24).

Thus we have obtained the following strategic game $\Gamma_B^* = \langle J = \{1,2,3,4\}, R_j, U_j \rangle$, where utility functions determined by the relations (24)-(27). Now, according to the relation (19) from the definition 6, we can determine the equilibrium profile. Strategy profiles $(\widetilde{x}_1^*, \widetilde{x}_2^*, \widetilde{y}_1^*, \widetilde{y}_2^*) \in NE(\Gamma_B^*)$ if and only if the following conditions are fulfilled

$$\begin{cases} U_1(\widetilde{x}_1^*, \widetilde{y}_1^*, \widetilde{y}_2^*, q) \geqslant U_1(\widetilde{x}_1, \widetilde{y}_1^*, \widetilde{y}_2^*, q) \text{ for all } \widetilde{x}_1 \in \widetilde{X}_1, \\ U_2(\widetilde{x}_2^*, \widetilde{y}_1^*, \widetilde{y}_2^*, q) \geqslant U_2(\widetilde{x}_2, \widetilde{y}_1^*, \widetilde{y}_2^*, q) \text{ for all } \widetilde{x}_2 \in \widetilde{X}_2, \\ U_3(\widetilde{x}_1^*, \widetilde{x}_2^*, \widetilde{y}_1^*, p) \geqslant U_3(\widetilde{x}_1^*, \widetilde{x}_2^*, \widetilde{y}_1, p) \text{ for all } \widetilde{y}_1 \in \widetilde{Y}_1, \\ U_4(\widetilde{x}_1^*, \widetilde{x}_2^*, \widetilde{y}_2^*, p) \geqslant U_4(\widetilde{x}_1^*, \widetilde{x}_2^*, \widetilde{y}_2, p) \text{ for all } \widetilde{y}_2 \in \widetilde{Y}_2. \end{cases}$$

Using the "best response approach", these relations are equivalent to the following

$$\begin{cases} \widetilde{x}_1^* \in Br_1(\widetilde{y}_1^*, \widetilde{y}_2^*, q), \\ \widetilde{x}_2^* \in Br_2(\widetilde{y}_1^*, \widetilde{y}_2^*, q), \\ \widetilde{y}_1^* \in Br_3(\widetilde{x}_1^*, \widetilde{x}_2^*, p), \\ \widetilde{y}_2^* \in Br_4(\widetilde{x}_1^*, \widetilde{x}_2^*, p), \end{cases} \tag{28}$$

where $Br_1(\widetilde{y}_1^*, \widetilde{y}_2^*, q) = Arg \max\limits_{\widetilde{x}_1 \in \widetilde{X}_1} U_1(\widetilde{x}_1, \widetilde{y}_1, \widetilde{y}_2, q)$, $Br_2(\widetilde{y}_1^*, \widetilde{y}_2^*, q) = Arg \max\limits_{\widetilde{x}_2 \in \widetilde{X}_2} U_2(\widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2, q)$, $Br_3(\widetilde{x}_1^*, \widetilde{x}_2^*, p) = Arg \max\limits_{\widetilde{y}_1 \in \widetilde{Y}_1} U_3(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_1, p)$ and $Br_4(\widetilde{x}_1^*, \widetilde{x}_2^*, p) = Arg \max\limits_{\widetilde{y}_2 \in \widetilde{Y}_2} U_4(\widetilde{x}_1, \widetilde{x}_2, \widetilde{y}_2, p)$. The relation (28) is equivalent to the following system

$$\begin{cases} \dfrac{\partial U_1(\widetilde{x}_1^*,\widetilde{y}_1^*,\widetilde{y}_2^*,q)}{\partial \widetilde{x}_1}=0, \\ \dfrac{\partial U_2(\widetilde{x}_2^*,\widetilde{y}_1^*,\widetilde{y}_2^*,q)}{\partial \widetilde{x}_2}=0, \\ \dfrac{\partial U_3(\widetilde{x}_1^*,\widetilde{x}_2^*,\widetilde{y}_1^*,p)}{\partial \widetilde{y}_1}=0, \\ \dfrac{\partial U_4(\widetilde{x}_1^*,\widetilde{x}_2^*,\widetilde{y}_2^*,p)}{\partial \widetilde{y}_2}=0, \end{cases} \Rightarrow \begin{cases} q\left[\dfrac{3}{2}\widetilde{y}_1^*-2\widetilde{x}_1^*\right]+(1-q)\left[\dfrac{3}{2}\widetilde{y}_2^*-2\widetilde{x}_1^*\right]=0, \\ q\left[\dfrac{3}{2}\widetilde{y}_1^*-2\widetilde{x}_2^*\right]+(1-q)\left[\dfrac{3}{2}\widetilde{y}_2^*-2\widetilde{x}_2^*\right]=0, \\ p\left[\dfrac{3}{2}\widetilde{x}_1^*-2\widetilde{y}_1^*\right]+(1-p)\left[\dfrac{3}{2}\widetilde{x}_2^*-2\widetilde{y}_1^*\right]=0, \\ p\left[\dfrac{3}{2}\widetilde{x}_1^*-2\widetilde{y}_2^*\right]+(1-p)\left[\dfrac{3}{2}\widetilde{x}_2^*-2\widetilde{y}_2^*\right]=0. \end{cases}$$

This system is equivalent to the following system

$$\begin{cases} \widetilde{x}_1^*=\dfrac{3}{4}\left[q\widetilde{y}_1^*+(1-q)\widetilde{y}_2^*\right]\in\left[0,\dfrac{3}{4}\right], \\ \widetilde{x}_2^*=\dfrac{3}{4}\left[q\widetilde{y}_1^*+(1-q)\widetilde{y}_2^*\right]\in[0,1], \\ \widetilde{y}_1^*=\dfrac{3}{4}\left[p\widetilde{x}_1^*+(1-p)\widetilde{x}_2^*\right]\in\left[0,\dfrac{3}{4}\right], \\ \widetilde{y}_2^*=\dfrac{3}{4}\left[p\widetilde{x}_1^*+(1-p)\widetilde{x}_2^*\right]\in[0,1]. \end{cases} \tag{29}$$

According to the Definition 8, for all $0\le p\le 1, 0\le q\le 1$, informational non extended Bayes-Nash equilibrium profiles $(x^*(q),y^*(p))$ are those for which the following conditions are fulfilled:

$$\begin{cases} \widetilde{x}_1^*=\theta_1^1(y)=\frac{3}{4}\left[q\theta_2^1(x)+(1-q)\theta_2^2(x)\right], \\ \widetilde{x}_2^*=\theta_1^2(y)=\frac{3}{4}\left[q\theta_2^1(x)+(1-q)\theta_2^2(x)\right], \\ \widetilde{y}_1^*=\theta_2^1(x)=\frac{3}{4}\left[p\theta_1^1(y)+(1-p)\theta_1^2(y)\right], \\ \widetilde{y}_2^*=\theta_2^2(x)=\frac{3}{4}\left[p\theta_1^1(y)+(1-p)\theta_1^2(y)\right]. \end{cases}$$

These relationships are obtained if in (29) we introduce the information extended strategies. In the particular case, if the player 1 chooses the informational extended strategy $\theta_1^1(y)=\frac{3}{4}y$ and assume with probability q that player 2 chooses the informational extended strategy $\theta_2^1(x)=\frac{3}{4}x$, and with probability $1-q$ the informational extended strategy $\theta_2^2(x)=x^2$, and, respectively[1], if the player 2 chooses the informational extended strategy $\theta_2^1(x)=\frac{3}{4}x$ and assume with probability p that player 1 chooses the informational extended strategy $\theta_1^1(y)=\frac{3}{4}y$ and with probability $1-p$ the informational extended strategy $\theta_1^2(y)=y^2$, then the informational non extended Bayes-Nash equilibrium profiles $(x^*(q),y^*(p))$ is calculated from the following system $\begin{cases} \theta_1^1(y)=\frac{3}{4}\left[q\theta_2^1(x)+(1-q)\theta_2^2(x)\right] \\ \theta_2^1(x)=\frac{3}{4}\left[p\theta_1^1(y)+(1-p)\theta_1^2(y)\right] \end{cases}$. So we have the system $\begin{cases} \frac{3}{4}y=\frac{3}{4}\left[q\frac{3}{4}x+(1-q)x^2\right] \\ \frac{3}{4}x=\frac{3}{4}\left[p\frac{3}{4}y+(1-p)\frac{3}{4}x\right] \end{cases}$. Finally, all solutions of this example are described in the following table.

Players type	Informational extended strategies	Solutions from system
$(1,1)=\left[(1,\theta_1^1),(2,\theta_2^1)\right]$	$\left(\theta_1^1(y),\theta_2^1(y)\right)=\left(\frac{3}{4}y,\frac{3}{4}x\right)$	$\begin{cases} y=\left[q\frac{3}{4}x+(1-q)x^2\right] \\ x=\left[p\frac{3}{4}y+(1-p)\frac{3}{4}x\right] \end{cases}$
$(2,1)=\left[(2,\theta_1^2),(2,\theta_2^1)\right]$	$\left(\theta_1^2(y),\theta_2^1(y)\right)=\left(y^2,\frac{3}{4}x\right)$	$\begin{cases} y^2=\frac{3}{4}\left[q\frac{3}{4}x+(1-q)x^2\right] \\ x=\left[p\frac{3}{4}y+(1-p)\frac{3}{4}x\right] \end{cases}$
$(1,2)=\left[(1,\theta_1^1),(2,\theta_2^2)\right]$	$\left(\theta_1^1(y),\theta_2^2(y)\right)=\left(\frac{3}{4}y,x^2\right)$	$\begin{cases} y=\left[q\frac{3}{4}x+(1-q)x^2\right] \\ x=\left[p\frac{3}{4}y+(1-p)\frac{3}{4}x\right] \end{cases}$
$(2,2)=\left[(2,\theta_1^2),(2,\theta_2^2)\right]$	$\left(\theta_1^2(y),\theta_2^2(y)\right)=\left(y^2,x^2\right)$	$\begin{cases} y^2=\frac{3}{4}\left[q\frac{3}{4}x+(1-q)x^2\right] \\ x^2=\frac{3}{4}\left[p\frac{3}{4}y+(1-p)y^2\right] \end{cases}$

[1]It is analyzed this case too, because the game is based on bidirectional information flows.

With this we finished solving the example.

For all $j=1,2,k=1,2$, denote by ϑ_1^j (respectively ϑ_2^k) an inverse function of the θ_1^j (respectively θ_2^k). If informational extended strategies $\theta_1^j(y)$ and $\theta_2^k(x)$ for all $j=1,2,k=1,2$ are bijective, then there are the inverse functions ϑ_1^j and ϑ_2^k for all $j=1,2,k=1,2$ and, so, we have that for all $\left(\left\{\tilde{x}_j^*\right\}_{l\in J_1},\{\tilde{y}_k^*\}_{k\in J_2}\right)\in NE\left[\Gamma_B^*\right]$ there is (x^*,y^*) such that $y^*=\vartheta_1^j(\tilde{x}_j^*)\ \forall j\in J_1$ and $x^*=\vartheta_1^k(\tilde{y}_j^*)\ \forall k\in J_2$.

According to [4] we can proof the following theorem.

Theorem 10 *Let the game* Γ *satisfy the following conditions:*

1) X and Y are a non-empty compact and convex subsets of the finite-dimensional Euclidean space;
2) the functions $\theta_1^j,\forall j\in J_1$, and θ_2^k, $\forall k\in J_2$, are continuous on Y (respectively on X) and the functions H_1,H_2 are continuous on $X\times Y$;
3) the functions $\theta_1^j,\forall j\in J_1$, (respectively θ_2^k, $\forall k\in J_2$), are quasi-concave on Y (respectively on X), the functions H_1 (respectively H_2) are quasi-concave on X (respectively Y) and monotonically increasing on $X\times Y$.

Then $BN\left[G\left(1\leftrightarrows 2\right)\right]\neq\emptyset$.

Proof. Another, and some times more convenient way of defining Nash equilibrium in the game Γ_B^* is via the best response correspondences that are defined as following. For all $j\in J_1$, $Br_j:\prod\limits_{k\in J_2}\widetilde{Y}_k\to 2^{\widetilde{X}_j}$,

$$Br_j\left(\{\widetilde{y}_k\}_{k\in J_2}\right)=\left\{\widetilde{x}_j\in\widetilde{X}_j:U_j\left(\widetilde{x}_j,\{\widetilde{y}_k\}_{k\in J_2}\right)\geq U_j\left(\widetilde{x}'_j,\{\widetilde{y}_k\}_{k\in J_2}\right)\text{ for all }\widetilde{x}'_j\in\widetilde{X}_j\right\}.$$

For all $j\in J_2$, $Br_j:\prod\limits_{k\in J_1}\widetilde{X}_k\to 2^{\widetilde{Y}_j}$,

$$Br_j\left(\{\widetilde{x}_k\}_{k\in J_1}\right)=\left\{\widetilde{y}_j\in\widetilde{Y}_j:U_j\left(\{\widetilde{x}_k\}_{k\in J_1},\widetilde{y}_j\right)\geq U_j\left(\{\widetilde{x}_k\}_{k\in J_1},\widetilde{y}'_j\right)\text{ for all }\widetilde{y}'_j\in\widetilde{Y}_j\right\}.$$

If we define the following point-to-set mapping $Br:\left(\prod_{k\in J_1}\widetilde{X}_k\right)\times\left(\prod_{k\in J_2}\widetilde{Y}_k\right)\to\left(\prod_{k\in J_1}\widetilde{X}_k\right)\times\left(\prod_{k\in J_2}\widetilde{Y}_k\right)$ by $Br\left(\{\widetilde{x}_k\}_{k\in J_1},\{\widetilde{y}_k\}_{k\in J_2}\right)=\left(\left\{Br_j\left(\{\widetilde{y}_k\}_{k\in J_2}\right)\right\}_{j\in J_1},\left\{Br_j\left(\{\widetilde{x}_k\}_{k\in J_1}\right)\right\}_{j\in J_2}\right)$, and if $\left(\left\{\widetilde{x}_j^*\right\}_{l\in J_1},\{\widetilde{y}_k^*\}_{k\in J_2}\right)\in Br\left(\left\{\widetilde{x}_j^*\right\}_{l\in J_1},\{\widetilde{y}_k^*\}_{k\in J_2}\right)$, then $\left(\left\{\widetilde{x}_j^*\right\}_{j\in J_1},\{\widetilde{y}_k^*\}_{k\in J_2}\right)\in NE\left[\Gamma_B^*\right]$. Here $\widetilde{x}_j=\theta_1^j(y)\ \forall j\in J_1$ and $\widetilde{y}_k=\theta_2^k(x)\ \forall k\in J_2$. Denote by $\widetilde{X}=\prod_{k\in J_1}\widetilde{X}_k$ and by $\widetilde{Y}=\prod_{k\in J_2}\widetilde{Y}_k$. The graph of the point-to-set mapping Br is

$$grBr=\left\{\left[\left(\{\widetilde{x}_k\}_{k\in J_1},\{\widetilde{y}_k\}_{k\in J_2}\right),\left(\left\{\widetilde{\widetilde{x}}_k\right\}_{k\in J_1},\left\{\widetilde{\widetilde{y}}_k\right\}_{k\in J_2}\right)\right]\big/\left(\{\widetilde{x}_k\}_{k\in J_1},\{\widetilde{y}_k\}_{k\in J_2}\right)\in\widetilde{X}\times\widetilde{Y},\right.$$
$$\left.\left(\left\{\widetilde{\widetilde{x}}_k\right\}_{k\in J_1},\left\{\widetilde{\widetilde{y}}_k\right\}_{k\in J_2}\right)\in Br\left(\{\widetilde{x}_k\}_{k\in J_1},\{\widetilde{y}_k\}_{k\in J_2}\right)\right\}.$$

So to prove this theorem we can show that: a) the $\prod_{k\in J_1}\widetilde{X}_k$ and $\prod_{k\in J_2}\widetilde{Y}_k$ are a non-empty compact and convex subsets of the Euclidean finite-dimensional space and b) the set-valued mapping $Br:\widetilde{X}\times\widetilde{Y}\to\widetilde{X}\times\widetilde{Y}$ has a closed graph, that is, if $\left\{\left(\{\widetilde{x}_k^l\}_{k\in J_1},\{\widetilde{y}_k^l\}_{k\in J_2}\right)\right\}_l\to\left(\{\widetilde{x}_k\}_{k\in J_1},\{\widetilde{y}_k\}_{k\in J_2}\right)$ and $\left\{\left(\left\{\widetilde{\widetilde{x}}_k^l\right\}_{k\in J_1},\left\{\widetilde{\widetilde{y}}_k^l\right\}_{k\in J_2}\right)\right\}_l\to\left(\left\{\widetilde{\widetilde{x}}_k\right\}_{k\in J_1},\left\{\widetilde{\widetilde{y}}_k\right\}_{k\in J_2}\right)$, where $\left(\left\{\widetilde{\widetilde{x}}_k^l\right\}_{k\in J_1},\left\{\widetilde{\widetilde{y}}_k^l\right\}_{k\in J_2}\right)\in$ $\in Br\left(\{\widetilde{x}_k^l\}_{k\in J_1},\{\widetilde{y}_k^l\}_{k\in J_2}\right)$ then $\left(\left\{\widetilde{\widetilde{x}}_k\right\}_{k\in J_1},\left\{\widetilde{\widetilde{y}}_k\right\}_{k\in J_2}\right)\in Br\left(\left(\{\widetilde{x}_k\}_{k\in J_1},\{\widetilde{y}_k\}_{k\in J_2}\right)\right)$, and the set

$Br\left(\left(\{\widetilde{x}_k\}_{k\in J_1}, \{\widetilde{y}_k\}_{k\in J_2}\right)\right)$ is nonempty, convex and compact for all $\left(\{\widetilde{x}_k\}_{k\in J_1}, \{\widetilde{y}_k\}_{k\in J_2}\right) \in \widetilde{X} \times \widetilde{Y}$. Since $\widetilde{x}_j = \theta_1^j(y)\ \forall j \in J_1$ and $\widetilde{y}_k = \theta_2^k(x)\ \forall k \in J_2$, then, according to condition 2), the sets $\widetilde{X}_k$ $\forall k \in J_1$ and $\widetilde{Y}_k\ \forall k \in J_2$ are compacts and, according to the Tikhonov's theorem: a product of a family of compact topological spaces is compact, the item a) is fulfilled. For all $\{\widetilde{y}_k\}_{k\in J_2}$ and $\{\widetilde{x}_k\}_{k\in J_1}$ the sets $Br_j\left(\{\widetilde{y}_k\}_{k\in J_2}\right)$ and $Br_j\left(\{\widetilde{x}_k\}_{k\in J_1}\right)$ are non-empty because of to conditions 1) and 2). According to condition 3), $Br_j\left(\{\widetilde{y}_k\}_{k\in J_2}\right)$ and $Br_j\left(\{\widetilde{x}_k\}_{k\in J_1}\right)$ are also convex sets. Hence the set $Br\left(\{\widetilde{x}_k\}_{k\in J_1}, \{\widetilde{y}_k\}_{k\in J_2}\right)$ is nonempty convex and compact for all $\widetilde{x}_k \in \widetilde{X}_k$, $k \in J_1$ and $\widetilde{y}_k \in \widetilde{Y}_k$, $k \in J_1$. According to condition 2) the mapping Br has a closed graph. The **Kakutani's theorem states** [10]: *Let S be a non-empty, compact and convex subset of the Euclidean space R^n. Let $\varphi : S \to 2^S$ be a set-valued function on S with a closed graph and the property that $\varphi(x)$ is non-empty and convex for all $x \in S$. Then φ has a fixed point.* Hence by Kakutani's theorem, the set-valued mapping Br has a fixed point. As we have noted, any fixed point is a Nash equilibrium. From the continuity of the functions $\theta_1^j, \forall j \in J_1$ and θ_2^k, $\forall k \in J_2$ it results that there exist inverse functions ϑ_1^j and ϑ_2^k for all $j = 1, 2, k = 1.2$ and so, we have that for all $\left(\left\{\tilde{x}_j^*\right\}_{l\in J_1}, \{\tilde{y}_k^*\}_{k\in J_2}\right) \in NE\left[\Gamma_B^*\right]$ there is (x^*, y^*) such that $y^* = \vartheta_1^j(\tilde{x}_j^*)\ \forall j \in J_1$ and $x^* = \vartheta_1^k(\tilde{y}_j^*)\ \forall k \in J_2$. The theorem is completely proved. ■

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Bibliography

[1] Hancu, B. Informational extended games generated by the one and double way directional informational flow. In: Studia Universitatis, Seria "Ştiinţe exacte şi economice", 2011, no. 7(47), p. 32–43.

[2] Novac, L. Nash equilibria in the noncooperative informational extended games. Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2009, no. 1(59), p. 96–103.

[3] Kukushkin N. S., Morozov V. V. *Teoria neantagonisticeskih igr.* Moskva: MGU, 1984(in Russian).

[4] Hancu, B. Solving the games generated by the informational extended strategies of the players. In: Buletinul Academiei de Stiinte a Republicii Moldova. Matematica. 2012, no. 3(70), p. 53-62.

[5] Rasmusen, E. Games and Information: An Introduction to Game Theory. Black-well Publishers. 2006.

[6] Harsanyi, John C. Games with incomplete information played by Bayesian players. Part I: The basic model. In: Management Science, 1967, no. 14, pp 159-182.

[7] Harsanyi, John C. Games with incomplete information played by Bayesian players. Part II: Bayesian equilibrium points. In: Management Science, 1968, no. 14, p. 320-334.

[8] Harsanyi, John C. Games with incomplete information played by Bayesian players. Part III: The basic probability distribution of the game. In: Management Science, 1968, no. 14, p. 486-502.

[9] Harsanyi, John C., Selten, R. *A General Theory of Equilibrium Selection in Games.* Cambridge, MA: MIT-Press, 1998.

[10] Kakutani, S. A generalization of Brouwer's fixed point theorem. In: Duke Mathematical Journal, 1941, no. 8(3), p. 457–459.