ON MULTIPLICATION GROUPS OF ISOSTROPHIC QUASIGROUPS

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A loop $(Q,\cdot)$ is called a middle Bol loop if every loop isotope of $(Q,\cdot)$ satisfies the identity $(x\cdot y)^{-1}=y^{-1}\cdot x^{-1}$ (i.e. if the anti-autmorphic inverse property is universal in $(Q,\cdot)$). Middle Bol loops are isostrophes of left (right) Bol loops. Multiplication groups of a quasigroup $(Q,\cdot)$ and of loops which are isostrophic to $(Q,\cdot)$ are characterized. In particular, it is proved that the right multiplication group of a middle Bol loop coincides with the left (right) multiplication group of the corresponding right (left) Bol loop. Some properties of the stabilizer of an element $h\in Q$ in the right (left) multiplication group of isostrophic quasigroups are established.
Keywords: *Bol loop, middle Bol loop, isostrophy, universal properties, multiplication groups, stabilizer.*

ASUPRA GRUPURILOR MULTIPLICATIVE ALE CVASIGRUPURILOR IZOSTROFE

Bucla $(Q,\cdot)$ se numește buclă medie Bol dacă în orice buclă izotopă cu $(Q,\cdot)$ are loc identitatea $(x\cdot y)^{-1}=y^{-1}\cdot x^{-1}$ (adică, dacă proprietatea anti-automorfică de inversabilitate este universală în bucla $(Q,\cdot)$). Buclele medii Bol sunt izostrofi ai buclelor Bol la stânga (dreapta). În lucrare sunt caracterizate grupurile multiplicative ale unui cvasigrup $(Q,\cdot)$ și ale buclelor care sunt izostrofe cu $(Q,\cdot)$. În particular, se demonstrează că grupul multiplicativ la dreapta al unei bucle medii Bol coincide cu grupul multiplicativ la stânga (dreapta) al buclei Bol la dreapta (stânga) corespunzătoare buclei $(Q,\cdot)$. Sunt stabilite un șir de proprietăți ale stabilizatorului unui element $h\in Q$ în grupurile multiplicative la dreapta (stânga) ale cvasigrupurilor izostrofe.
Cuvinte-cheie: *buclă Bo, buclă medie Bol, izostrofie, proprietăți universale, grupuri multiplicative, stabilizator.*

Introduction

A grupoid $(Q,\cdot)$ is called a quasigroup if the equations $a\cdot x=b$ and $y\cdot a=b$ have unique solutions in Q, for $\forall a,b\in Q$. Two quasigroups $(Q,\cdot)$ and $(Q,*)$ are isotopic (and we say that $(Q,*)$ is an isotope of $(Q,\cdot)$) if there exist three one-to-one mappings $\alpha,\beta,\gamma\in S_Q$, such that $x*y=\gamma^{-1}(\alpha x\cdot\beta y)$, $\forall x,y\in Q$. The triple (α,β,γ) is called an isotopy of the quasigroup $(Q,\cdot)$ and we'll denote $(*)=(\cdot)^{(\alpha,\beta,\gamma)}$. If $(*)=(\cdot)$ then (α,β,γ) is called an autotopism of $(Q,\cdot)$. If (Q,A) is a quasigroup and $\sigma\in S_3$. The operation ${}^{\sigma}A$, defintied by the equivalence

$${}^{\sigma}A\left(x_{\sigma(1)},x_{\sigma(2)}\right)=x_{\sigma(3)}\Leftrightarrow A(x_1,x_2)=x_3$$

is called a parastrophe of the operation A. The isotopes of a parastrophe of the quasigroup (Q,A) are called isotrophes of (Q,A). A quasigroup with a neutral element is called a loops. The loops satisfying the identity $(xy\cdot z)y=x(yz\cdot y)$ (respectively $x(y\cdot xz)=(x\cdot yx)z$) are called left Bol loops (respectively, right Bol loops) [1, 3, 6]. The theory of left (right) Bol loops was developed in a series of works by *Bol G., Robinson D.A., Pflugfelder H.O., R.P. Burn, E.G. Goodaire, Nagy G.P., Kiechle H.,* and others.

The loops satisfying one of the identities $x(yz\cdot x)=xy\cdot zx$, $z(x\cdot zy)=(zx\cdot z)y$, or $x(z\cdot yz)=(xz\cdot y)z$ (witch are equivalent in loops) are called Moufang loops.

A loop $(Q,\cdot)$ is called a middle Bol loop if it satisfies the following identity

$$x\cdot(yz\backslash x)=(x/z)\cdot(y\backslash x),$$

where „\", („/") is the right division (respectively, left division) in the loop $(Q,\cdot)$. Middle Bol loops were defined by V. Belousov in [1]. Moreover, the last identity is a necessary and sufficient condition for the universality of the anti-automorphic inverse property$(x\cdot y)^{-1}=y^{-1}\cdot x^{-1}$. A. Gwaramija proved in [2] that a loop $(Q,\circ)$ is middle Bol if and only if there exists a right Bol loop $(Q,\cdot)$ such that

$$x\circ y=(y\cdot xy^{-1})y\,, \tag{1}$$

for every $x,y\in Q$. From (1) follows $(y\cdot x)\circ y^{-1}=[y^{-1}\cdot(yx\cdot y)]\cdot y^{-1}=xy\cdot y^{-1}=x$, for every $x,y\in Q$, so we have

$$y\cdot x=x//y^{-1}\,, \tag{2}$$

for every $x,y\in Q$, where "$//$" is the left division in $(Q,\circ)$. On the other hand, denoting $x\cdot y=z$, $x=y\backslash x$, from (2) follows: $z=(y\backslash z)//y^{-1}\Leftrightarrow z\circ y^{-1}=y\backslash z$, so

$$z\circ y=y^{-1}\backslash z, \tag{3}$$

for every $x, y \in Q$. Hence, the loops $(Q,\cdot)$ and $(Q,\circ)$ are isostrophic. Also, it is shown in [4] that a loop $(Q,\circ)$ is middle Bol if and only if there exists a left Bol loop $(Q,\cdot)$ such that

$$x \circ y = y(y^{-1}x \cdot y),$$

for every $x, y \in Q$. From the last equality follows:

$$x \cdot y = x//y^{-1}, \quad (4)$$

$$x \circ y = x/y^{-1}, \quad (5)$$

for every $x, y \in Q$. Other properties of middle Bol lop are studied in [2,4,5].

Let $(Q,\cdot)$ be a quasigroup. Denote by $L_a^{(\cdot)}$ $(resp. R_a^{(\cdot)})$ the left (resp., right) translation with the element a in $(Q,\cdot)$. We will use below the notations:
$LM(Q,\cdot) = \langle L_x | x \in Q\rangle$ – the left multiplication group of $(Q,\cdot)$,
$RM(Q,\cdot) = \langle R_x | x \in Q\rangle$ – the right multiplication group of $(Q,\cdot)$,
$M(Q,\cdot) = \langle L_x, R_x | x \in Q\rangle$ – the multiplication group of $(Q,\cdot)$,
where $L_a^{(\cdot)}(y) = a \cdot y$, $R_a^{(\cdot)}(y) = y \cdot a$, $\forall a, y \in Q$.

Remark 1. It is proved in [5] that two middle Bol loops are isotopic if and only if the corresponding right (left) Bol loops are isotopic. Relations between multiplication groups of a quasigroup and those of its principal isotopes are found by K. Sciukhin in [7].

Theorem 1. [7] *Let the loop $(Q,\circ)$ be a principal izotope, of a quasigroup $(Q,\cdot)$, with isotopy $x \circ y = \varphi(x) \cdot \psi(y)$. Then the following statements hold:*

1. $LM(Q,\circ) = \langle L_x^{(\cdot)}\psi | x \in Q\rangle = \langle L_x^{(\cdot)-1} L_y^{(\cdot)} | x, y \in Q\rangle$;
2. $RM(Q,\circ) = \langle R_y^{(\cdot)}\varphi | y \in Q\rangle = \langle R_y^{(\cdot)-1} R_x^{(\cdot)} | x, y \in Q\rangle$;
3. $M(Q,\circ) = \langle L_x^{(\cdot)}\psi,\ R_y^{(\cdot)}\varphi | x, y \in Q\rangle = \langle L_x^{(\cdot)-1} L_y^{(\cdot)}, R_u^{(\cdot)-1} R_v^{(\cdot)} | x, y, u, v \in Q\rangle$;
4. *If* $\psi \in Aut(Q,\circ)$ *then* $LM(Q,\circ) \lhd LM(Q,\cdot)$;
5. *If* $\psi \in Aut(Q,\circ)$ *then* $RM(Q,\circ) \lhd RM(Q,\cdot)$;
6. *If* $\psi \in Aut(Q,\circ)$ *then* $M(Q,\circ) \lhd M(Q,\cdot)$;

Let $(Q,\cdot)$ *be a quasigroup and* $x \in Q$, *will use below the mapping* $I_x^{(\cdot)}: Q \to Q$ $\quad I_x^{(\cdot)}(y) = y\backslash x$, $\forall y \in Q$. Remark that $I_x^{(\cdot)}$ is a bijection, and that $I_x^{(\cdot)-1}(y) = x/y$, $\forall y \in Q$.

Proposition 1. [8] *Let $(Q,\cdot)$ be a quasigroup and $\varphi, \psi \in S_Q$, such that the isostroph $(Q,\circ)$, where $x \circ y = \psi(y)\backslash\varphi(x)$, $\forall x, y \in Q$, is a loop . The following statements hold:*

1. $LM(Q,\circ) = \langle I_x^{(\cdot)}\psi | x \in Q\rangle$;
2. $RM(Q,\circ) = \langle L_x^{(\cdot)-1}\varphi | x \in Q\rangle = \langle L_x^{(\cdot)-1} L_y^{(\cdot)} | x, y \in Q\rangle \leq LM(Q,\cdot)$;
3. $M(Q,\circ) = \langle I_x^{(\cdot)}\psi,\ L_y^{(\cdot)}\varphi | x, y \in Q\rangle = \langle I_x^{(\cdot)}\psi, L_y^{(\cdot)-1} L_z^{(\cdot)} | x, y, z \in Q\rangle$;
4. *If φ is an automorphism of $(Q,\circ)$ then* $RM(Q,\circ) \trianglelefteq LM(Q,\cdot)$;
5. $LM(Q,\cdot) = \langle RM(Q,\circ), \varphi\rangle$.

Corollary 1. [8] If $(Q,\circ)$ is a middle Bol loop and $(Q,\cdot)$ be the corresponding right Bol loop. Then the following equalities are true:
$RM(Q,\circ) = LM(Q,\cdot)$.

Proposition 2. *Let $(Q,\cdot)$ be a quasigroup and $\varphi, \psi \in S_Q$ such that the isostroph $(Q,\circ)$, where $x \circ y = \varphi(x)/\psi(y)$, $\forall x, y \in Q$, is a loop. The following statements hold:*

1. $LM(Q,\circ) = \langle I_x^{(\cdot)-1}\psi | x \in Q\rangle$;
2. $RM(Q,\circ) = \langle R_y^{(\cdot)-1}\varphi | y \in Q\rangle = \langle R_y^{(\cdot)-1} R_x^{(\cdot)} | x, y \in Q\rangle \leq RM(Q,\cdot)$;
3. $M(Q,\circ) = \langle I_x^{(\cdot)-1}\psi,\ R_y^{(\cdot)-1}\varphi | x, y \in Q\rangle = \langle I_x^{(\cdot)-1}\psi, R_y^{(\cdot)-1} R_z^{(\cdot)} | x, y, z \in Q\rangle$;
4. $RM(Q,\circ) \lhd RM(Q,\cdot)$, *if φ is an automorphism of $(Q,\circ)$*;
5. $RM(Q,\cdot) = \langle RM(Q,\circ), \varphi\rangle$.

Proof. 1. According to the definition, $x \circ y = \varphi(x)/\psi(y)$, which implies $L_x^{(\circ)}(y) = I_{\varphi(x)}^{(\cdot)-1}\psi(y)$, so
$LM(Q,\circ) = \langle I_{\varphi(x)}^{(\cdot)-1}\psi | x \in Q\rangle$.

2. Let $e \in Q$ be the unit of the loop $(Q,\circ)$. Then $x = x \circ e = \varphi(x)/\psi(e) \Rightarrow \varphi(x) = x \cdot \psi(e)$ so
$R_{\psi(e)}^{(\cdot)}(x) = \varphi(x)$, $\forall x \in Q$,i.e $\varphi = R_{\psi(e)}^{(\cdot)}$.

Hence, from $x\circ y=\varphi(x)/\psi(y)$ follows $R_y^{(\circ)}(x)=R_{\psi(y)}^{(\cdot)-1}\varphi(x)$, $\forall x\in Q$, so on the other hand

$$R_y^{(\circ)}=R_{\psi(y)}^{(\cdot)-1}\varphi=R_{\psi(y)}^{(\cdot)-1}R_{\psi(e)}^{(\cdot)}. \qquad (6)$$

In particular, we get that, for $\forall y\in Q$, the following equality holds:

$$R_y^{(\cdot)-1}=R_{\psi^{-1}(y)}^{(\circ)}\varphi^{-1}. \qquad (7)$$

Using (6), we have:

$$RM(Q,\circ)=\langle R_y^{(\circ)}|y\in Q\rangle=\langle R_x^{(\cdot)-1}\varphi|x\in Q\rangle\subseteq\langle R_x^{(\cdot)-1}R_y^{(\cdot)}|x,y\in Q\rangle.$$

On the other hand,

$$R_x^{(\cdot)-1}R_y^{(\cdot)}=R_{\psi(\psi^{-1}(x))}^{(\cdot)-1}R_{\psi(e)}^{(\cdot)}R_{\psi(e)}^{(\cdot)-1}R_{\psi(\psi^{-1}(y))}^{(\cdot)}=R_{\psi^{-1}(x)}^{(\circ)}R_{\psi^{-1}(y)}^{(\circ)-1}\in RM(Q,\circ),$$

so

$$RM(Q,\circ)=\langle R_x^{(\cdot)-1}R_y^{(\cdot)}|x,y\in Q\rangle.$$

3. Follows from 1 and 2.

4. Let φ be an automorphism of $(Q,\circ)$, then $\varphi(x\circ y)=\varphi(x)\circ\varphi(y)$, for every $x,y\in Q$, so $\varphi R_y^{(\circ)}(x)=R_{\varphi(y)}^{(\circ)}\varphi(x)$, $\forall x\in Q$, hence $\varphi R_y^{(\circ)}=R_{\varphi(y)}^{(\circ)}\varphi$, $\forall y\in Q$, and

$$\varphi R_y^{(\circ)}\varphi^{-1}=R_{\varphi(y)}^{(\circ)}, \qquad (8)$$

for every $y\in Q$. Using (7) and (8), let's show that for every $R_x^{(\cdot)}\in RM(Q,\cdot)$ and every $R_y^{(\circ)}\in RM(Q,\circ)$, we have $R_x^{(\cdot)}R_y^{(\circ)}R_x^{(\cdot)-1}\in RM(Q,\circ)$:

$$R_x^{(\cdot)}R_y^{(\circ)}R_x^{(\cdot)-1}=\varphi R_{\psi^{-1}(x)}^{(\circ)-1}R_y^{(\circ)}R_{\psi^{-1}(x)}^{(\circ)}\varphi^{-1}=\varphi R_{\psi^{-1}(x)}^{(\circ)-1}\varphi^{-1}\varphi R_y^{(\circ)}\varphi^{-1}\varphi R_{\psi^{-1}(x)}^{(\circ)}\varphi^{-1}=$$
$$=R_{\varphi(\psi^{-1}(x))}^{(\circ)-1}R_{\varphi(y)}^{(\circ)}R_{\varphi(\psi^{-1}(x))}^{(\circ)}\in RM(Q,\circ).$$

Analogously, using (7) and (8) we'll prove that $R_x^{(\cdot)-1}R_y^{(\circ)}R_x^{(\cdot)}\in RM(Q,\circ)$:

$$R_x^{(\cdot)-1}R_y^{(\circ)}R_x^{(\cdot)}=R_{\psi^{-1}(x)}^{(\circ)}\varphi^{-1}R_y^{(\circ)}\varphi R_{\psi^{-1}(x)}^{(\circ)-1}=R_{\psi^{-1}(x)}^{(\circ)-1}R_{\varphi^{-1}(y)}^{(\circ)}R_{\psi^{-1}(x)}^{(\circ)}\in RM(Q,\circ).$$

So as

$$R_x^{(\cdot)}R_y^{(\circ)-1}R_x^{(\cdot)-1}=\left(R_x^{(\cdot)}R_y^{(\circ)}R_x^{(\cdot)-1}\right)^{-1}=\left(R_{\varphi(\psi^{-1}(x))}^{(\circ)-1}R_{\varphi(y)}^{(\circ)}R_{\varphi(\psi^{-1}(x))}^{(\circ)}\right)^{-1}\in RM(Q,\circ)$$

and

$$R_x^{(\cdot)-1}R_y^{(\circ)-1}R_x^{(\cdot)}=\left(L_x^{(\cdot)-1}R_y^{(\circ)}L_x^{(\cdot)}\right)^{-1}=\left(R_{\psi^{-1}(x)}^{(\circ)-1}R_{\varphi^{-1}(y)}^{(\circ)}R_{\psi^{-1}(x)}^{(\circ)}\right)^{-1}\in RM(Q,\circ),$$

we get that: $\delta R_y^{(\circ)}\delta^{-1}$, $\delta^{-1}R_y^{(\circ)}\delta$, $\delta R_y^{(\circ)-1}\delta^{-1}$, $\delta^{-1}R_y^{(\circ)-1}\delta\in RM(Q,\circ)$, $\forall\delta\in RM(Q,\cdot)$. So, we proved that $RM(Q,\circ)\lhd RM(Q,\cdot)$.

5. Using (7), we have $R_y^{(\cdot)}=\varphi R_{\psi^{-1}(y)}^{(\circ)-1}\in\langle RM(Q,\circ),\varphi\rangle$, so $RM(Q,\cdot)\subseteq\langle RM(Q,\circ),\varphi\rangle$. So as $\varphi=R_{\psi(e)}^{(\cdot)}\in\langle R_x^{(\cdot)}|x\in Q\rangle$ (see the proof of 2.) and $R_x^{(\circ)}=R_{\psi(x)}^{(\cdot)-1}R_{\psi(e)}^{(\cdot)}\in\langle R_x^{(\cdot)}|x\in Q\rangle$, we get that $\langle RM(Q,\circ),\varphi\rangle\subseteq RM(Q,\cdot)$, so $RM(Q,\cdot)=\langle RM(Q,\circ),\varphi\rangle$. □

Corollary 2. Let $(Q,\circ)$ be a middle Bol loop, and let $(Q,\cdot)$ be the corresponding left Bol loop. Then
$RM(Q,\circ)=RM(Q,\cdot)$.

Proof. According to (4) $x\cdot y=x//y^{-1}=x//I(y)$, for every $x,y\in Q$, where $I:Q\to Q, I(x)=x^{-1}$. From the Proposition 2, p. 2, for $\varphi=\varepsilon$, and $\psi=I$, we obtain $RM(Q,\circ)=\langle R_y^{(\cdot)-1}|y\in Q\rangle=RM(Q,\cdot)$. □

Proposition 3. *Let $(Q,\cdot)$ be a quasigroup and $\varphi,\psi\in S_Q$, such that the isostroph $(Q,\circ)$, where $x\circ y=\varphi(x)\backslash\psi(y)$, $\forall x,y\in Q$, is a loop. The following statements hold:*

1. $RM(Q,\circ)=\langle I_y^{(\cdot)}\varphi|y\in Q\rangle$;
2. $LM(Q,\circ)=\langle L_y^{(\cdot)-1}\psi|y\in Q\rangle=\langle L_x^{(\cdot)-1}L_y^{(\cdot)}|x,y\in Q\rangle\leq LM(Q,\cdot)$;
3. $M(Q,\circ)=\langle I_y^{(\cdot)}\varphi,\ L_x^{(\cdot)-1}\psi|x,y\in Q\rangle=\langle I_z^{(\cdot)}\varphi, L_x^{(\cdot)-1}L_y^{(\cdot)}|x,y,z\in Q\rangle$;
4. *If ψ is an automorphism of $(Q,\circ)$ then $LM(Q,\circ)\lhd LM(Q,\cdot)$;*
5. $LM(Q,\cdot)=\langle LM(Q,\circ),\psi\rangle$.

Proof. 1. By the definition, $\circ y=\varphi(x)\backslash\psi(y)$, which implies $R_y^{(\circ)}(x)=I_{\psi(y)}^{(\cdot)}\varphi(x)$, so $RM(Q,\circ)=\langle I_{\psi(y)}^{(\cdot)}\varphi|y\in Q\rangle$.

2. Let $e \in Q$ be the unit of the loop $(Q,\circ)$. Then $y = e \circ y = \varphi(e)\backslash\psi(y) \Rightarrow \psi(y) = \varphi(e) \cdot y$ so $L^{(\cdot)}_{\varphi(e)}(y) = \psi(y)$, $\forall y \in Q$, i.e. $\psi = L^{(\cdot)}_{\varphi(e)}$. Hence, from $x \circ y = \varphi(x)\backslash\psi(y)$ follows $L^{(\circ)}_x(y) = L^{(\cdot)-1}_{\varphi(x)}\psi(y)$, $\forall x \in Q$, which implies

$$L^{(\circ)}_x(y) = L^{(\cdot)-1}_{\varphi(x)}\psi = L^{(\cdot)-1}_{\varphi(x)}L^{(\cdot)}_{\varphi(e)}. \quad (9)$$

In particular, we get that, for $x \in Q$, the following equality holds:

$$L^{(\cdot)-1}_x = L^{(\circ)}_{\varphi^{-1}(x)}\psi^{-1}\ . \quad (10)$$

Using (9) and the equality $\psi = L^{(\cdot)}_{\varphi(e)}$ we have:

$$LM(Q,\circ) = \langle L^{(\circ)}_y | y \in Q\rangle = \langle L^{(\cdot)-1}_y \psi | y \in Q\rangle \subseteq \langle L^{(\cdot)-1}_x L^{(\cdot)}_y | x, y \in Q\rangle.$$

On the other hand,

$$L^{(\cdot)-1}_x L^{(\cdot)}_y = L^{(\cdot)-1}_{\varphi(\varphi^{-1}(x))} L^{(\cdot)}_{\varphi(e)} L^{(\cdot)-1}_{\varphi(e)} L^{(\cdot)}_{\varphi(\varphi^{-1}(y))} = L^{(\circ)}_{\varphi^{-1}(x)} L^{(\circ)-1}_{\varphi^{-1}(y)} \in LM(Q,\circ),$$

so

$$LM(Q,\circ) = \langle L^{(\cdot)-1}_x L^{(\cdot)}_y | x, y \in Q\rangle.$$

3. Follows from 1. and 2.
4. Let ψ be an automorphism of $(Q,\circ)$, then $\psi(x \circ y) = \psi(x) \circ \psi(y)$, for every $x, y \in Q$. The last equality implies: $L^{(\circ)}_x(y) = L^{(\circ)}_{\psi(x)}\psi(y)$, $\forall x \in Q$, hence $\psi L^{(\circ)}_x = L^{(\circ)}_{\psi(x)}\psi$, $\forall x \in Q$, and

$$\psi L^{(\circ)}_x \psi^{-1} = L^{(\circ)}_{\psi(x)} \quad (11)$$

for every $x \in Q$. Using (10) and (11), let's show that for every $L^{(\cdot)}_x \in LM(Q,\cdot)$ and $L^{(\circ)}_y \in LM(Q,\circ)$, to show that $L^{(\cdot)}_x L^{(\circ)}_y L^{(\cdot)-1}_x \in LM(Q,\circ)$:

$$L^{(\cdot)}_x L^{(\circ)}_y L^{(\cdot)-1}_x = \psi L^{(\circ)-1}_{\varphi^{-1}(x)} L^{(\circ)}_y L^{(\circ)}_{\varphi^{-1}(x)} \psi^{-1} = \psi L^{(\circ)-1}_{\varphi^{-1}(x)} \psi^{-1} \psi L^{(\circ)}_y L^{(\circ)}_{\varphi^{-1}(x)} \psi^{-1}\psi\psi^{-1} =$$

$$= L^{(\circ)-1}_{\psi(\varphi^{-1}(x))} L^{(\circ)}_{\psi(x)} L^{(\circ)}_{\psi(\varphi^{-1}(x))} \in LM(Q,\circ).$$

Analogously, using (10) and (11) we'll prove that $L^{(\cdot)-1}_x L^{(\circ)}_y L^{(\cdot)}_x \in LM(Q,\circ)$:

$$L^{(\cdot)-1}_x L^{(\circ)}_y L^{(\cdot)}_x = L^{(\circ)}_{\varphi^{-1}(x)} \psi^{-1} L^{(\circ)}_y \psi L^{(\circ)-1}_{\varphi^{-1}(x)} = L^{(\circ)}_{\varphi^{-1}(x)} L^{(\circ)}_{\varphi^{-1}(y)} L^{(\circ)-1}_{\varphi^{-1}(x)} \in RM(Q,\circ).$$

So as

$$L^{(\cdot)}_x L^{(\circ)-1}_y L^{(\cdot)-1}_x = \left(L^{(\cdot)}_x L^{(\circ)}_y L^{(\cdot)-1}_x\right)^{-1} = \left(L^{(\circ)-1}_{\psi(\varphi^{-1}(x))} L^{(\circ)}_{\psi(x)} L^{(\circ)}_{\psi(\varphi^{-1}(x))}\right)^{-1} \in LM(Q,\circ)$$

and

$$L^{(\cdot)-1}_x L^{(\circ)-1}_y L^{(\cdot)}_x = \left(L^{(\cdot)-1}_x L^{(\circ)}_y L^{(\cdot)}_x\right)^{-1} = \left(L^{(\circ)}_{\varphi^{-1}(x)} L^{(\circ)}_{\varphi^{-1}(y)} L^{(\circ)-1}_{\varphi^{-1}(x)}\right)^{-1} \in LM(Q,\circ).$$

We get: $\delta L^{(\circ)}_y \delta^{-1}$, $\delta^{-1} L^{(\circ)}_y \delta$, $\delta L^{(\circ)-1}_y \delta^{-1}$, $\delta^{-1} L^{(\circ)-1}_y \delta \in LM(Q,\circ)$, $\forall \delta \in LM(Q,\cdot)$; $LM(Q,\circ) \trianglelefteq LM(Q,\cdot)$.
5. Follows from 2. . □

Proposition 4. *Let $(Q,\cdot)$ be a quasigroup and $\varphi, \psi \in S_Q$ such that the isostroph $(Q,\circ)$, where $x \circ y = \psi(y)/\varphi(x)$, $\forall x, y \in Q$, is a loop. The following statements hold:*

1. $RM(Q,\circ) = \langle I^{(\cdot)-1}_y \varphi | y \in Q\rangle$;
2. $LM(Q,\circ) = \langle R^{(\cdot)-1}_y \psi | y \in Q\rangle = \langle R^{(\cdot)-1}_y R^{(\cdot)}_z | y, z \in Q\rangle \leq RM(Q,\cdot)$;
3. $M(Q,\circ) = \langle I^{(\cdot)-1}_y \varphi,\ R^{(\cdot)-1}_x \psi | x, y \in Q\rangle = \langle I^{(\cdot)-1}_z \varphi, R^{(\cdot)-1}_x R^{(\cdot)}_y | x, y, z \in Q\rangle$;
4. *If ψ is an automorphism of then $(Q,\circ)LM(Q,\circ) \lhd RM(Q,\cdot)$;*
5. $RM(Q,\cdot) = \langle LM(Q,\circ), \psi\rangle$.

Proof. 1. According to the definition $x \circ y = \psi(y)/\varphi(x)$, which implies $R^{(\circ)}_y(x) = I^{(\cdot)-1}_{\psi(y)}\varphi(x)$, so obtain $RM(Q,\circ) = \langle I^{(\cdot)-1}_{\psi(y)}\varphi | y \in Q\rangle$.
2. Let $e \in Q$ be the unit of the loop $(Q,\circ)$. Then $y = e \circ y = \psi(y)/\varphi(e) \Rightarrow \psi(y) = y \cdot \varphi(e) \Rightarrow R^{(\cdot)}_{\varphi(e)}(y) = \psi(y)$, $\forall y \in Q$, i.e $\psi = R^{(\cdot)}_{\varphi(e)}$. Hence, from $x \circ y = \psi(y)/\varphi(x)$ follows $L^{(\circ)}_x(y) = R^{(\cdot)-1}_{\varphi(x)}\psi(y)$, $\forall x, y \in Q$, which implies

$$L^{(\circ)}_x(y) = R^{(\cdot)-1}_{\varphi(x)}\psi = R^{(\cdot)-1}_{\varphi(x)} R^{(\cdot)}_{\varphi(e)}. \quad (12)$$

In particular, we get that, for $x \in Q$, the following equality holds:

$$R_x^{(\cdot)-1} = L_{\varphi^{-1}(x)}^{(\circ)}\psi^{-1}\,. \tag{13}$$

Using (12), we have:

$$LM(Q,\circ) = \langle L_y^{(\circ)} | y \in Q\rangle = \langle R_y^{(\cdot)-1}\psi | y \in Q\rangle \subseteq \langle R_x^{(\cdot)-1}R_y^{(\cdot)} | x, y \in Q\rangle.$$

On the other hand,

$$R_x^{(\cdot)-1}R_y^{(\cdot)} = R_{\varphi(\varphi^{-1}(x))}^{(\cdot)-1}R_{\varphi(e)}^{(\cdot)}R_{\varphi(e)}^{(\cdot)-1}R_{\varphi(\varphi^{-1}(y))}^{(\cdot)} = L_{\varphi^{-1}(x)}^{(\circ)}L_{\varphi^{-1}(y)}^{(\circ)-1} \in LM(Q,\circ)$$

so

$$LM(Q,\circ) = \langle R_x^{(\cdot)-1}R_y^{(\cdot)} | x, y \in Q\rangle.$$

3. Follows from 1 and 2.
4. Let ψ be an automorphism of $(Q,\circ)$, using (13) and (11), let's show that for every $R_x^{(\cdot)} \in LM(Q,\cdot)$ and every $L_y^{(\circ)} \in RM(Q,\circ)$, we have $R_x^{(\cdot)}L_y^{(\circ)}R_x^{(\cdot)-1} \in LM(Q,\circ)$:

$$R_x^{(\cdot)}L_y^{(\circ)}R_x^{(\cdot)-1} = \psi L_{\varphi^{-1}(x)}^{(\circ)-1}L_y^{(\circ)}L_{\varphi^{-1}(x)}^{(\circ)}\psi^{-1} = \psi L_{\varphi^{-1}(x)}^{(\circ)-1}\psi^{-1}\psi L_y^{(\circ)}\psi^{-1}\psi L_{\varphi^{-1}(x)}^{(\circ)}\psi^{-1} =$$
$$= L_{\psi(\varphi^{-1}(x))}^{(\circ)-1}L_{\psi(x)}^{(\circ)}L_{\psi(\varphi^{-1}(x))}^{(\circ)} \in LM(Q,\circ).$$

Analogously, using (13) and (11) we'll prove that $R_x^{(\cdot)}L_y^{(\circ)}R_x^{(\cdot)-1} \in LM(Q,\circ)$:

$$R_x^{(\cdot)-1}L_y^{(\circ)}R_x^{(\cdot)} = L_{\varphi^{-1}(x)}^{(\circ)}\psi^{-1}L_y^{(\circ)}\psi L_{\varphi^{-1}(x)}^{(\circ)-1} = L_{\varphi^{-1}(x)}^{(\circ)}L_{\psi^{-1}(y)}^{(\circ)}L_{\varphi^{-1}(x)}^{(\circ)-1} \in LM(Q,\circ).$$

So as

$$R_x^{(\cdot)}L_y^{(\circ)-1}R_x^{(\cdot)-1} = \left(R_x^{(\cdot)}L_y^{(\circ)}R_x^{(\cdot)-1}\right)^{-1} = \left(L_{\psi(\varphi^{-1}(x))}^{(\circ)-1}L_{\psi(x)}^{(\circ)}L_{\psi(\varphi^{-1}(x))}^{(\circ)}\right)^{-1} \in LM(Q,\circ)$$

and

$$R_x^{(\cdot)-1}L_y^{(\circ)-1}R_x^{(\cdot)} = \left(R_x^{(\cdot)-1}L_y^{(\circ)-1}R_x^{(\cdot)}\right)^{-1} = \left(L_{\varphi^{-1}(x)}^{(\circ)}L_{\psi^{-1}(y)}^{(\circ)}L_{\varphi^{-1}(x)}^{(\circ)-1}\right)^{-1} \in LM(Q,\circ),$$

we get that: $\delta L_y^{(\circ)}\delta^{-1}$, $\delta^{-1}L_y^{(\circ)}\delta$, $\delta L_y^{(\circ)-1}\delta^{-1}$, $\delta^{-1}L_y^{(\circ)-1}\delta \in LM(Q,\circ)$, $\forall \delta \in LM(Q,\cdot)$, i.e. we proved that $LM(Q,\circ) \lhd RM(Q,\cdot)$.
Follows from 2. □

Let $(Q,\cdot)$ be a qusigroup $h \in Q$. The set $M(Q,\cdot)_h = \{\varphi \in M(Q,\cdot) | \varphi(h) = h\}$, i.e. the stabilizer of h in $M(Q,\cdot)$, is called the group of inner mappings, with respect to h, of $(Q,\cdot)$, and will be denoted by $I_h^{(\cdot)}$. Below we'll use also the notations
$RM(Q,\cdot)_h = \{\varphi \in RM(Q,\cdot) | \varphi(h) = h\}$ for the stabilizer of h in $RM(Q,\cdot)$ and
$(LM)_h^{(\cdot)} = \{\varphi \in LM(Q,\cdot) | \varphi(h) = h\}$ for the stabilizer of h in $LM(Q,\cdot)$.

Proposition 5. Let $(Q,\cdot)$ be a quasigroup, $h \in Q$ and $\varphi, \psi \in S_Q$. If $(Q,\circ)$ is isostroph of $(Q,\cdot)$ given by the isostrophy $\circ\, y = \psi(y)\backslash\varphi(x)$, $\forall x, y \in Q$, then $RM(Q,\circ)_h = RM(Q,\circ) \cap LM(Q,\cdot)_h$.
Proof. Let $\alpha \in RM(Q,\circ)_h$. Using Proposition 1 we get

$$\alpha \in RM(Q,\circ) \le LM(Q,\cdot) \text{ and } \alpha(h) = h,$$

so $\alpha \in RM(Q,\circ) \cap LM(Q,\cdot)_h$ i. e.

$$RM(Q,\circ)_h \subseteq RM(Q,\circ) \cap LM(Q,\cdot)_h. \tag{14}$$

Conversely, if $\alpha \in RM(Q,\circ) \cap LM(Q,\cdot)_h$, then $\alpha \in RM(Q,\circ)$ and $\alpha(h) = h$, i. e.

$$RM(Q,\circ) \cap LM(Q,\cdot)_h \subseteq RM(Q,\circ)_h. \tag{15}$$

From (14) and (15) follows the equality $RM(Q,\circ)_h = RM(Q,\circ) \cap LM(Q,\cdot)_h$. □

Corollary 3. Let $(Q,\cdot)$ be a right Bol loop and let $(Q,\circ)$ be the corresponding middle Bol loop of $(Q,\cdot)$. Then, for every $h \in Q$, the following equality holds

$$RM(Q,\circ)_h = LM(Q,\cdot)_h$$

Proof. The proof follows from Corollary 1 and Proposition 5. □

Proposition 6. Let $(Q,\cdot)$ be a quasigroup, $h \in Q$ and $\varphi, \psi \in S_Q$. If $(Q,\circ)$ is the isostroph of $(Q,\cdot)$ given by $x \circ y = \varphi(x) / \psi(y)$, $x, y \in Q$, then $RM(Q,\circ)_h = RM(Q,\circ) \cap RM(Q,\cdot)_h$.
Proof. Let $RM(Q,\circ)_h$. Using Proposition 2 we get

$$\alpha \in RM(Q,\circ) \le RM(Q,\cdot) \text{ and } \alpha(h) = h,$$

so $\alpha \in RM(Q,\circ) \cap RM(Q,\cdot)_h$, i. e.

$$RM(Q,\circ)_h \subseteq RM(Q,\circ) \cap LM(Q,\cdot)_h. \quad (16)$$

Conversely, if $\alpha \in RM(Q,\circ) \cap RM(Q,\cdot)_h$, then $\alpha \in RM(Q,\circ)$ and $\alpha(h) = h$ i. e.

$$RM(Q,\circ) \cap RM(Q,\cdot)_h \subseteq RM(Q,\circ)_h. \quad (17)$$

From (16) and (17) follows the equality $RM(Q,\circ)_h = RM(Q,\circ) \cap RM(Q,\cdot)_h$. □

Corollary 4. If $(Q,\cdot)$ is a left Bol loop and $(Q,\circ)$ is the corresponding middle Bol loop of $(Q,\cdot)$ then

$$(RM)_h^{(\circ)} = (RM)_h^{(\cdot)}.$$

Proof. The proof follows from Corollary 1 and Proposition 5. □

Proposition 7. Let $(Q,\cdot)$ be a quasigroup, $h \in Q$ and $\varphi, \psi \in S_Q$. If $(Q,\circ)$ is the isostroph of $(Q,\cdot)$ given by $x \circ y = \varphi(x) \backslash \psi(y)$, $\forall x, y \in Q$, then $LM(Q,\circ)_h = LM(Q,\circ) \cap LM(Q,\cdot)_h$.

Proof. Let $\alpha \in LM(Q,\circ)_h$. Using Proposition 3, we get:

$$\alpha \in LM(Q,\circ) \leq LM(Q,\cdot) \text{ and } \alpha(h) = h,$$

so $\alpha \in LM(Q,\circ) \cap LM(Q,\cdot)_h$, i. e.

$$LM(Q,\circ)_h \subseteq LM(Q,\circ) \cap LM(Q,\cdot)_h. \quad (18)$$

Conversely, if $\alpha \in LM(Q,\circ) \cap LM(Q,\cdot)_h$, then $\alpha \in LM(Q,\circ)$ and $\alpha(h) = h$ i. e.

$$LM(Q,\circ) \cap LM(Q,\cdot)_h \subseteq LM(Q,\circ)_h. \quad (19)$$

From (18) and (19) follows the equality $LM(Q,\circ)_h = LM(Q,\circ) \cap LM(Q,\cdot)_h$. □

Proposition 8. Let $(Q,\cdot)$ be a quasigroup, $h \in Q$ and $\varphi, \psi \in S_Q$. If $(Q,\circ)$ is isostroph of $(Q,\cdot)$ given by the isostrophy $x \circ y = \psi(y)/\varphi(x)$, $\forall x, y \in Q$, then $LM(Q,\circ)_h = LM(Q,\circ) \cap RM(Q,\cdot)_h$.

Proof. Let $\alpha \in LM(Q,\circ)_h$. Using Proposition 4 we get:

$$\alpha \in LM(Q,\circ) \leq RM(Q,\cdot) \text{ and } \alpha(h) = h,$$

so $\alpha \in LM(Q,\circ) \cap RM(Q,\cdot)_h$. i e.

$$LM(Q,\circ)_h \subseteq LM(Q,\circ) \cap RM(Q,\cdot)_h. \quad (20)$$

Conversely, if $\alpha \in LM(Q,\circ) \cap RM(Q,\cdot)_h$, then $\alpha \in LM(Q,\circ)$ and $\alpha(h) = h$ i. e.

$$LM(Q,\circ) \cap RM(Q,\cdot)_h \subseteq LM(Q,\circ)_h. \quad (21)$$

From (20) and (21) follows the equality $LM(Q,\circ)_h = LM(Q,\circ) \cap RM(Q,\cdot)_h$. □

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