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ON PSEUDOAUTOMORPHISMS OF MIDDLE BOL LOOPS

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A loop $(Q,\cdot)$ is called a middle Bol loop if every loop isotope of $(Q,\cdot)$ satisfies the identity $(x\cdot y)^{-1}=y^{-1}x^{-1}$ (i.e. if the anti-automorphic inverse property is universal in $(Q,\cdot)$) [1]. Middle Bol loops are isostrophes of left (right) Bol loops [2, 4]. The left (right, middle) pseudoautomorphisms of middle Bol loops are considered in the present article. The general form of middle Bol loop's autotopisms is given using right pseudoautomorphisms of the corresponding right Bol loops. Necessary and sufficient conditions when a LP-isotope of a middle Bol loop $(Q,\cdot)$ is isomorphic to $(Q,\cdot)$ are proved. It is shown that in the left (right) Bol loops every middle pseudoautomorphism is a left (right) pseudoautomorphism. Connections between the groups of pseudoautomorphisms (left, right, middle) of a middle Bol loop and of the corresponding left Bol loop are found.
Keywords: *middle Bol loop, middle (left, right) pseudoautomorphism, autotopy, isostrophy.*

ASUPRA PSEUDOAUTOMORFISMELOR BUCLELOR MEDII BOL

O buclă $(Q,\cdot)$ se numeşte buclă medie Bol, dacă orice buclă izotopă cu $(Q,\cdot)$ verifică identitatea $(x\cdot y)^{-1}=y^{-1}x^{-1}$ (dacă proprietatea antiautomorfică de inversabilitate este universală în $(Q,\cdot)$) [1]. Buclele medii Bol sunt izostrofi ai buclelor Bol la stânga (la dreapta) [2]. În prezentul articol sunt studiate pseudoautomorfismele la stânga (la dreapta, medii). Este dedusă forma generală a autotopiilor buclelor medii Bol cu ajutorul pseudoautomorfismelor la dreapta ale buclelor Bol la dreapta corespunzătoare. Sunt date condiții necesare şi suficiente ca un LP-izotop al unei bucle medii Bol $(Q,\cdot)$ să fie izomorf cu $(Q,\cdot)$. Se demonstrează că în buclele Bol la stânga (la dreapta) orice pseudoautomofism mediu este un pseudoautomorfism la stânga (la dreapta). Sunt stabilite conexiuni între grupurile de pseudoautomorfisme (la stânga, la dreapta, medii) ale unei bucle medii Bol şi cele ale buclei Bol la stânga corespunzătoare.
Cuvinte-chee: *buclă medie Bol, pseudoautomorfism mediu (la dreapta, la stânga), autotopie, izostrofie.*

A loop $(Q,\cdot)$ is called a right (left) Bol loop if it satisfies the identity $(zx\cdot y)x=z(xy\cdot x)$ ($x(y\cdot xz)=(x\cdot yx)z$). Right (left) Bol loops are studied, for example, in [3, 7]. A loop $(Q,\cdot)$ is called a middle Bol loop if the identity $(x\cdot y)^{-1}=y^{-1}x^{-1}$ (the anti-automorphic inverse property) is universal in $(Q,\cdot)$, i.e. is invariant under loop isotopy. It is shown in [1] that a loop $(Q,\cdot)$ is middle Bol if and only if the corresponding primitive loop $(Q,\cdot,/,\backslash)$ satisfies the middle Bol identity: $x(yz\backslash x)=(x/z)(y\backslash x)$. Middle Bol loops are studied in [1, 2, 4, 5]. A.Gwaramija proved in [2] that middle Bol loops are isostrophes of right (left) Bol loops: a loop $(Q,\circ)$ is middle Bol if and only if there exists a right Bol loop $(Q,\cdot)$ such that

$$x\circ y=y^{-1}\backslash x, \tag{1}$$

where „\" is the right division in $(Q,\cdot)$. From (1) follows

$$y\cdot x=x//y^{-1}, \tag{2}$$

for every $x,y\in Q$, where „//" is the left division in $(Q,\circ)$. If $(Q,\cdot)$ is the corresponding left Bol loop of $(Q,\circ)$, then

$$x\circ y=x/y^{-1}, \tag{3}$$

$$x\cdot y=x//y^{-1}, \tag{4}$$

for every $x,y\in Q$, where „/" („//") is the left division in $(Q,\cdot)$ (resp. $(Q,\circ)$).

The left (right, middle) pseudoautomorphisms of middle Bol loops are considered in the present article. The general form of middle Bol loop's autotopisms is given using right pseudoautomorphisms of the corresponding right Bol loops. Necessary and sufficient conditions when a LP-isotope of a middle Bol loop $(Q,\cdot)$ is isomorphic to $(Q,\cdot)$ are proved. It is shown that in the left (right) Bol loops every middle pseudoautomorphism is a left (resp. right) pseudoautomorphism. Connections between the groups of pseudoautomorphisms (left, right, middle) of a middle Bol loop and of the corresponding left Bol loop are found.

Let $(Q,\cdot)$ be a middle Bol loop and $a \in Q$. Consider the bijection: $I_a: Q \to Q$, $I_a(x) = x \backslash a$, then $I_a^{-1}: Q \to Q$, $I_a^{-1}(x) = a/x$, $\forall a, x \in Q$. In particular, if $(Q,\cdot)$ is a commutative middle Bol loop then $I_a = I_a^{-1}$, $\forall a \in Q$. Hence, using the middle Bol identity, we get that a loop $(Q,\cdot)$ is a middle Bol loop if and only if the triple $(I_x^{-1}I, I_xI, L_xI_xI)$ is an autotopism of $(Q,\cdot)$, where $I: Q \to Q$, $I(x) = x^{-1}$, i.e. is the inversion in $(Q,\cdot)$.

Let $(Q,\cdot)$ be an arbitrary loop, $\varphi \in S_Q$ and $c \in Q$. Remind that: a) φ is called a *left (resp. right) pseudoautomorphism* of $(Q,\cdot)$, with the companion c, if the equality

$$c \cdot \varphi(x \cdot y) = [c \cdot \varphi(x)] \cdot \varphi(y) \text{ (resp., } \varphi(x \cdot y) \cdot c = \varphi(x) \cdot [\varphi(y) \cdot c])$$

holds, for every $x, y \in Q$; b) φ is called a *middle pseudoautomorphism,* with the companion c, if the equality

$$\varphi(x \cdot y) = [\varphi(x)/c^{-1}] \cdot [c \backslash \varphi(y)], \tag{5}$$

holds, for every $x, y \in Q$, where c^{-1} is the right inverse of c. The notion of pseudoautomophism was introduced by Bruck in [12] for IP-loops. The pseudoautomorphisms of LIP-loops has been studied by Florea ([8], [9]). In particular, Florea found the general form of autotopisms of LIP-loops and proved that a principal isotope of a LIP-loop $(Q,\cdot)$ is isomorphic to $(Q,\cdot)$ if and only if $(Q,\cdot)$ has a left pseudoautomophism with the companion $k = a \cdot ba$, where a is a Bol element and $b \in Q$. The notion of *middle pseudoautomorphism* was introduced by A. Drisco in [11] and also considered in [10]. Middle pseudoautomorphisms of (right) Bruck loops are studied in [6]. Below we will denote by $PS_r^{(\cdot)}$ (resp., $PS_l^{(\cdot)}$, $PS_m^{(\cdot)}$) the group of all right (resp. left, middle) pseudoautomorphisms of the loop $(Q,\cdot)$.

Remark 1. Let $(Q,\cdot)$ be an arbitrary loop. It is easy to see that any automorphism $\varphi \in Aut(Q,\cdot)$ is a middle pseudoautomorphism with the companion e, where e is the unit of $(Q,\cdot)$.

Lemma 1. *Let $(Q,\cdot)$ be an arbitrary loop, $\varphi \in S_Q$ and $c \in Q$. Then φ is a middle pseudoautomorphism with the companion c if and only if $\varphi^{-1}(x \cdot y) = \varphi^{-1}(x \cdot c^{-1}) \cdot \varphi^{-1}(c \cdot y)$, $\forall x, y \in Q$.*

Proof. Let φ be a middle pseudoautomorphism with the companion c, then (5) holds for every $x, y \in Q$. Denoting $c \backslash \varphi(y) = u$, $\varphi(x)/c^{-1} = v$, we have $c \cdot u = \varphi(y)$, $v \cdot c^{-1} = \varphi(x)$, so

$$y = \varphi^{-1}(c \cdot u),\ x = \varphi^{-1}(v \cdot c^{-1}). \tag{6}$$

From (5) and (6) we get:

$$\varphi(\varphi^{-1}(v \cdot c^{-1}) \cdot \varphi^{-1}(c \cdot u)) = v \cdot u \Leftrightarrow \varphi^{-1}(v \cdot u) = \varphi^{-1}(v \cdot c^{-1}) \cdot \varphi^{-1}(c \cdot u),$$

$\forall u, v \in Q$. □

Corollary. *Let $(Q,\cdot)$ be an arbitrary loop, $\varphi \in S_Q$* and *$c \in Q$. Then φ is a middle pseudoautomorphism with the companion c if and only if the triple $(\varphi^{-1}R_{c^{-1}}, \varphi^{-1}L_c, \varphi^{-1})$ is an autotopism of $(Q,\cdot)$.*

Proposition 1. *Let $(Q,\cdot)$ be a middle Bol loop with the unit e. If $x^2 = e$, for all $x \in Q$, then every $a \in Q$ is a companion for some middle pseudoautomorphism of $(Q,\cdot)$.*

Proof. Let $(Q,\cdot)$ be a middle Bol loop with $x^2 = e$, for all $x \in Q$, then $x = x^{-1}$ and $x \cdot y = (x \cdot y)^{-1} = y^{-1} \cdot x^{-1} = y \cdot x$, for all $x, y \in Q$, so $(Q,\cdot)$ is a commutative middle Bol loop. Let $a \in Q$, then the triple $(I_a^{-1}I, I_aI, L_aI_aI)$ is an autotopism of $(Q,\cdot)$, for all $a \in Q$. Denoting L_aI_aI by φ, we get that the triple $(I_a^{-1}I, I_aI, \varphi)$ is an autotopism of $(Q,\cdot)$, so

$$\varphi(x \cdot y) = I_a^{-1}I(x) \cdot I_aI(y), \tag{7}$$

for all $x, y \in Q$. Taking $y = e$ in (7), we obtain $\varphi(x) = I_a^{-1}I(x) \cdot I_a I(e) = I_a^{-1}I(x) \cdot a = R_a I_a^{-1}I(x)$, for all $x \in Q$, so

$$I_a^{-1}I = R_a^{-1}\varphi = R_{a^{-1}}^{-1}\varphi, \tag{8}$$

for all $a \in Q$. Now, taking $x = e$ in (7), we have $\varphi(y) = I_a^{-1}I(e) \cdot I_a I(y) = a \cdot I_a I(y) = L_a I_a I(y)$, for all $x \in Q$, hence

$$I_a I = L_a^{-1}\varphi, \tag{9}$$

for all $a \in Q$. Using (9) and (8), from (7) we get that the triple $(R_{a^{-1}}^{-1}\varphi, L_a^{-1}\varphi, \varphi)$ is an autotopism of $(Q,\cdot)$, so φ is a middle pseudoautomorphism of $(Q,\cdot)$ with the companion a. □

Let $(Q,\cdot)$ be an arbitrary loop, we will denote by $N_r^{(\cdot)}$ (resp. $N_l^{(\cdot)}$, $N_m^{(\cdot)}$) the right (resp. left, middle) nucleus of the loop $(Q,\cdot)$.

Proposition 2. *If $(Q,\cdot)$ is a middle Bol loop and $a \in N_m^{(\cdot)}$ then $R_a^{-1}I_a^{-1}I$ is a right pseudoautomorfism with the companion a^2.*

Proof. Let $(Q,\cdot)$ be a middle Bol loop and $a \in N_m^{(\cdot)}$. Then the triples $(I_a^{-1}I, I_a I, L_a I_a I)$ and $(R_a^{-1}, L_a, \varepsilon)$ are autotopisms of $(Q,\cdot)$, hence

$$(R_a^{-1}, L_a, \varepsilon) \cdot (I_a^{-1}I, I_a I, L_a I_a I) = (R_a^{-1}I_a^{-1}I, L_a I_a I, L_a I_a I)$$

is an autotopism of $(Q,\cdot)$. Denoting $R_a^{-1}I_a^{-1}I$ by τ in the triple $(R_a^{-1}I_a^{-1}I, L_a I_a I, L_a I_a I)$, we obtain:

$$L_a I_a I(x \cdot y) = \tau(x) \cdot L_a I_a I(y), \tag{10}$$

for all $x, y \in Q$, so taking $y = e$ in (10), where e is the unit of $(Q,\cdot)$, we get $L_a I_a I(x) = R_{a^2}\tau(x)$, for all $x \in Q$, and for all $a \in N_m^{(\cdot)}$, so the triple $(\tau, R_{a^2}\tau, R_{a^2}\tau)$ is an autotopism of $(Q,\cdot)$, i.e. τ is a pseudoautomorphism of $(Q,\cdot)$ with the companion a^2. □

Proposition 3. *Let $(Q,\cdot)$ be a middle Bol loop with the unit e and let $T = (\alpha, \beta, \gamma)$ be an autotopism of $(Q,\cdot)$. Then there exists a right pseudoautomorphism τ with the companion $k = b^{-1}\backslash a^{-1}$, where $a = \alpha(e)$, $b = \beta(e)$, such that*

$$T = (II_{a^{-1}}, II_{a^{-1}}^{-1}, II_{a^{-1}}^{-1}L_{a^{-1}}^{-1}) \cdot (\tau, R_k\tau, R_k\tau).$$

Proof. Let $(Q,\cdot)$ be a middle Bol loop, then the triple $(I_{c^{-1}}^{-1}I, I_{c^{-1}}I, L_{c^{-1}}I_{c^{-1}}I)$ is an autotopism of $(Q,\cdot)$, for all $c \in Q$. If $T = (\alpha, \beta, \gamma)$ is an autotopism of $(Q,\cdot)$, then the triple $T_1 = (I_{a^{-1}}^{-1}I, I_{a^{-1}}I, L_{a^{-1}}I_{a^{-1}}I) \cdot (\alpha, \beta, \gamma) = (I_{a^{-1}}^{-1}I\alpha, I_{a^{-1}}I\beta, L_{a^{-1}}I_{a^{-1}}I\gamma)$ is an autotopism of $(Q,\cdot)$. Denoting $I_{a^{-1}}^{-1}I\alpha = \tau$, we obtain $\tau(e) = I_{a^{-1}}^{-1}I\alpha(e) = I_{a^{-1}}^{-1}(a^{-1}) = a^{-1}/a^{-1} = e$, so

$$L_{a^{-1}}I_{a^{-1}}I\gamma(x \cdot y) = \tau(x) \cdot I_{a^{-1}}I\beta(y) \tag{11}$$

for all $x, y \in Q$. Taking $x = e$ in (11), we get $L_{a^{-1}}I_{a^{-1}}I\gamma(y) = I_{a^{-1}}I\beta(y)$, for all $y \in Q$, and using the equality $L_{a^{-1}}I_{a^{-1}}I\gamma = I_{a^{-1}}I\beta$, where $a \in Q$, we obtain that the triple $(\tau, I_{a^{-1}}I\beta, I_{a^{-1}}I\beta)$ is an autotopism of $(Q,\cdot)$, for all $a \in Q$. So, for $\forall x, y \in Q$, the equality $I_{a^{-1}}I\beta(x \cdot y) = \tau(x) \cdot I_{a^{-1}}I\beta(y)$ holds and, taking $y = e$ in the last equality, we obtain: $I_{a^{-1}}I\beta(x) = \tau(x) \cdot I_{a^{-1}}I\beta(e) \Rightarrow I_{a^{-1}}I\beta(x) = R_k\tau(x)$, for all $x \in Q$, where $k = I_{a^{-1}}I\beta(e) = b^{-1}\backslash a^{-1}$, hence $I_{a^{-1}}I\beta = R_k\tau$, i.e.
$T_1 = (\tau, I_{a^{-1}}I\beta, I_{a^{-1}}I\beta) = (\tau, R_k\tau, R_k\tau)$ is an autotopism of $(Q,\cdot)$ which implies that τ is a right pseudoautomophism with the companion k. So $T_1 = (I_{a^{-1}}^{-1}I, I_{a^{-1}}I, L_{a^{-1}}I_{a^{-1}}I) \cdot (\alpha, \beta, \gamma) = (\tau, R_k\tau, R_k\tau)$, i. e.

$$(\alpha, \beta, \gamma) = (II_{a^{-1}}, II_{a^{-1}}^{-1}, II_{a^{-1}}^{-1}L_{a^{-1}}^{-1}) \cdot (\tau, R_k\tau, R_k\tau),$$

where $k = b^{-1}\backslash a^{-1}$, $a = \alpha(e)$, $b = \beta(e)$. □

An analogous result was obtained by A. Gwaramija for middle Bol loops and their corresponding left Bol loops.

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Proposition 4 [2]. *Let $(Q,\cdot)$ be a middle Bol loop with the unit e and let $T = (\alpha, \beta, \gamma)$ be an autotopism of $(Q,\cdot)$. Then there exists a left pseudoautomorphism τ with the companion $k_1 = b^{-1}/a^{-1}$, where $a = \alpha(e), b = \beta(e)$, such that*

$$T = \left(IL_{b^{-1}}, IL_{b^{-1}}^{-1}, IL_{b^{-1}}^{-1}L_{b^{-1}}^{-1}\right)\cdot\left(L_{k_1}\tau, \tau, L_{k_1}\tau\right). \qquad (12)$$

Theorem 1. *Let $(Q,\cdot)$ be a middle Bol loop, $a, b \in Q$, and let $x \circ y = R_a^{-1}x \cdot L_b^{-1}y$, $\forall x, y \in Q$. Then $(Q,\circ) \cong (Q,\cdot)$ if and only if there exists a right pseudoautomorphism of $(Q,\cdot)$ with the companion $k = a^{-1}\backslash b^{-1}$.*

Proof. Let $(Q,\circ) \cong (Q,\cdot)$, and let γ be an isomorphism between them. Then $\gamma(x\cdot y) = \gamma(x)\circ\gamma(y) = R_a^{-1}\gamma(x)\cdot L_b^{-1}\gamma(y) = \beta(x)\cdot\alpha(y)$, for all $x, y \in Q$, where $\beta = R_a^{-1}\gamma$, $\alpha = L_b^{-1}\gamma$, so the triple (β, α, γ) is an autotopism of $(Q,\cdot)$ as well. Let 1 and e be the units of $(Q,\cdot)$ and $(Q,\circ)$, respectively. So as γ is an isomorphism we have $\gamma(1) = e$, where $e = b\cdot a$, and:

$$\beta(1) = R_a^{-1}\gamma(1) = R_a^{-1}(e) = R_a^{-1}(b\cdot a) = b, \qquad (13)$$

$$\alpha(1) = L_b^{-1}\gamma(1) = L_b^{-1}(e) = L_b^{-1}(b\cdot a) = a. \qquad (14)$$

From Proposition 3, (13) and (14) we obtain the equality

$$(\beta, \alpha, \gamma) = \left(IL_{b^{-1}}, IL_{b^{-1}}^{-1}, IL_{b^{-1}}^{-1}L_{b^{-1}}^{-1}\right)\cdot(\tau, R_k\tau, R_k\tau),$$

where τ is a right pseudoautomorphism with the companion $k = a^{-1}\backslash b^{-1}$.

Conversely, if τ is a right pseudoautomorphism of $(Q,\cdot)$ with the companion $k = a^{-1}\backslash b^{-1}$, then the triple $(\tau, R_k\tau, R_k\tau)$ is an autotopism of $(Q,\cdot)$. On the other hand, so as $(Q,\cdot)$ is a middle Bol loop we get that $\left(L_{b^{-1}}^{-1}I, L_{b^{-1}}I, L_{b^{-1}}L_{b^{-1}}I\right)^{-1} = \left(IL_{b^{-1}}, IL_{b^{-1}}^{-1}, IL_{b^{-1}}^{-1}L_{b^{-1}}^{-1}\right)$ is an autotopism of $(Q,\cdot)$. So,

$$(\beta, \alpha, \gamma) = \left(IL_{b^{-1}}, IL_{b^{-1}}^{-1}, IL_{b^{-1}}^{-1}L_{b^{-1}}^{-1}\right)\cdot(\tau, R_k\tau, R_k\tau) = \left(IL_{b^{-1}}\tau, IL_{b^{-1}}^{-1}R_k\tau, IL_{b^{-1}}^{-1}L_{b^{-1}}^{-1}R_k\tau\right),$$

is an autotopism of $(Q,\cdot)$, which implies

$$IL_{b^{-1}}^{-1}L_{b^{-1}}^{-1}R_k\tau(x\cdot y) = IL_{b^{-1}}\tau(x)\cdot IL_{b^{-1}}^{-1}R_k\tau(y) \Leftrightarrow \gamma(x\cdot y) = \beta(x)\cdot\alpha(y) \qquad (15)$$

for all $x, y \in Q$, where $\gamma = IL_{b^{-1}}^{-1}L_{b^{-1}}^{-1}R_k\tau$, $\beta = IL_{b^{-1}}\tau$, $\alpha = IL_{b^{-1}}^{-1}R_k\tau$. Also,

$$\beta(1) = IL_{b^{-1}}\tau(1) = IL_{b^{-1}}(1) = (b^{-1})^{-1} = b,$$

$$\alpha(1) = IL_{b^{-1}}^{-1}R_k\tau(1) = IL_{b^{-1}}^{-1}R_k(1) = \left(b^{-1}/(a^{-1}\backslash b^{-1})\right)^{-1} = (a^{-1})^{-1} = a$$

hence, denoting $b^{-1}/(a^{-1}\backslash b^{-1}) = z$, we get $z\cdot(a^{-1}\backslash b^{-1}) = b^{-1}$, and denoting $a^{-1}\backslash b^{-1} = c$, we obtain $z\cdot c = b^{-1}$, so $z\cdot c = a^{-1}\cdot c \Rightarrow z = a^{-1}$. Now, taking $y = 1$ in (15) we obtain $(x) = \beta(x)\cdot a \Rightarrow \gamma(x) = R_a\beta(x)$, for all $x \in Q$, i.e.

$$\beta = R_a^{-1}\gamma. \qquad (16)$$

Taking $x = 1$ in (15), we have: $\gamma(y) = b\cdot\alpha(y) \Rightarrow \gamma(y) = L_b\alpha(y)$, for all $y \in Q$, so

$$\alpha = L_b^{-1}\gamma. \qquad (17)$$

From (15), (16) and (17) follows: $\gamma(x\cdot y) = \beta(x)\cdot\alpha(y) = R_a^{-1}\gamma(x)\cdot L_b^{-1}\gamma(y) = \gamma(x)\circ\gamma(y)$, for all $x, y \in Q$, i.e. $(Q,\circ) \cong (Q,\cdot)$. □

Theorem 2. *Let $(Q,\cdot)$ be a middle Bol loop, $a, b \in Q$, and let $x \circ y = R_a^{-1}x \cdot L_b^{-1}y$, $\forall x, y \in Q$. Then $(Q,\circ) \cong (Q,\cdot)$ if and only if there exists a left pseudoautomorphism of $(Q,\cdot)$ with the companion $k_1 = a^{-1}/b^{-1}$.*

Proof. Let $(Q,\circ) \cong (Q,\cdot)$, and let γ be an isomorphism between them, then $\gamma(x\cdot y) = \gamma(x)\circ\gamma(y) = R_a^{-1}\gamma(x)\cdot L_b^{-1}\gamma(y) = \beta(x)\cdot\alpha(y)$, for all $x, y \in Q$, where $\beta = R_a^{-1}\gamma$,

$\alpha = L_b^{-1}\gamma$. So, (β, α, γ) is an autotopism of $(Q,\cdot)$. Let 1 and e be the units of $(Q,\cdot)$ and $(Q,\circ)$, respectively. So as γ is an isomorphism, $\gamma(1) = e$, where $e = b\cdot a$, and the following equalities hold:

$$\beta(1) = R_a^{-1}\gamma(1) = R_a^{-1}(e) = R_a^{-1}(b\cdot a) = b, \tag{18}$$

$$\alpha(1) = L_b^{-1}\gamma(1) = L_b^{-1}(e) = L_b^{-1}(b\cdot a) = a. \tag{19}$$

From Proposition 4, (18) and (19) we obtain that

$$(\beta,\alpha,\gamma) = \left(II_{a^{-1}}, II_{a^{-1}}^{-1}, II_{a^{-1}}^{-1}L_{a^{-1}}^{-1}\right)\cdot\left(L_{k_1}\tau, \tau, L_{k_1}\tau\right),$$

where τ is a left pseudoautomorphism of $(Q,\cdot)$, with the companion $k_1 = a^{-1}/b^{-1}$. Conversely, let τ be a left pseudoautomorphism of $(Q,\cdot)$, with the companion $k = a^{-1}/b^{-1}$. Then $(L_{k_1}\tau, \tau, L_{k_1}\tau)$ is an autotopism of $(Q,\cdot)$. So as $(Q,\cdot)$ is a middle Bol loop,

$$\left(I_{a^{-1}}^{-1}I, I_{a^{-1}}I, L_{a^{-1}}I_{a^{-1}}I\right)^{-1} = \left(II_{a^{-1}}, II_{a^{-1}}^{-1}, II_{a^{-1}}^{-1}L_{a^{-1}}^{-1}\right)$$

is an autotopism of $(Q,\cdot)$. So we get that

$$(\beta,\alpha,\gamma) = \left(II_{a^{-1}}, II_{a^{-1}}^{-1}, II_{a^{-1}}^{-1}L_{a^{-1}}^{-1}\right)\cdot\left(L_{k_1}\tau, \tau, L_{k_1}\tau\right) = \left(II_{a^{-1}}L_{k_1}\tau, II_{a^{-1}}^{-1}\tau, II_{a^{-1}}^{-1}L_{a^{-1}}^{-1}L_{k_1}\tau\right),$$

is an autotopism of $(Q,\cdot)$, and

$$II_{a^{-1}}^{-1}L_{a^{-1}}^{-1}L_{k_1}\tau(x\cdot y) = II_{a^{-1}}L_{k_1}\tau(x)\cdot II_{a^{-1}}^{-1}\tau(y) \Leftrightarrow \gamma(x\cdot y) = \beta(x)\cdot\alpha(y) \tag{20}$$

for all $x, y \in Q$, where $\gamma = II_{a^{-1}}^{-1}L_{a^{-1}}^{-1}L_{k_1}\tau$, $\beta = II_{a^{-1}}L_{k_1}\tau$, $\alpha = II_{a^{-1}}^{-1}\tau$. Also

$$\beta(1) = II_{a^{-1}}L_{k_1}\tau(1) = II_{a^{-1}}L_{k_1}(1) = II_{a^{-1}}(a^{-1}/b^{-1}) =$$

$$\left((a^{-1}/b^{-1})\backslash a^{-1}\right)^{-1} = (b^{-1})^{-1} = b.$$

Denoting $\left((a^{-1}/b^{-1})\backslash a^{-1}\right) = z$, we get $(a^{-1}/b^{-1})\cdot z = a^{-1}$, and taking $a^{-1}/b^{-1} = c$ we have $c\cdot b^{-1} = a^{-1}$, so $c\cdot z = c\cdot b^{-1}$ i.e. $z = b^{-1}$ and $\alpha(1) = II_{a^{-1}}^{-1}\tau(1) = II_{a^{-1}}^{-1}(1) = (a^{-1})^{-1} = a$. Now, for $y = 1$ in (20) we obtain $\gamma(x) = \beta(x)\cdot a$, which implies $\gamma(x) = R_a\beta(x)$, for all $x \in Q$, so

$$\beta = R_a^{-1}\gamma. \tag{21}$$

Taking $x = 1$ in (20) we have $\gamma(y) = b\cdot\alpha(y)$, so $\gamma(y) = L_b\alpha(y)$, for all $y \in Q$, i.e.

$$\alpha = L_b^{-1}\gamma. \tag{22}$$

From (20), (21) and (22) we get: $\gamma(x\cdot y) = \beta(x)\cdot\alpha(y) = R_a^{-1}\gamma(x)\cdot L_b^{-1}\gamma(y) = \gamma(x)\circ\gamma(y)$, for all $x, y \in Q$, so $(Q,\circ) \cong (Q,\cdot)$. □

Proposition 5. *Let $(Q,\cdot)$ be an arbitrary loop. The following statements hold:*

1. $\varphi\left(N_l^{(\cdot)}\right) = N_l^{(\cdot)}$ *and* $\varphi\left(N_m^{(\cdot)}\right) = N_m^{(\cdot)}$, *for every* $\varphi \in PS_r^{(\cdot)}$;
2. $\varphi\left(N_r^{(\cdot)}\right) = N_r^{(\cdot)}$ *and* $\varphi\left(N_m^{(\cdot)}\right) = N_m^{(\cdot)}$, *for every* $\varphi \in PS_l^{(\cdot)}$;

Proof. 1. Let $a \in N_l^{(\cdot)}$, and let φ be a *right pseudoautomorphism* of $(Q,\cdot)$ with the companion $c \in Q$. Then

$$\varphi(a)\cdot[\varphi(x)\cdot(\varphi(y)\cdot c)] = \varphi(a)\cdot(\varphi(x\cdot y)\cdot c) =$$

$$\varphi(a\cdot xy)\cdot c = \varphi(ax\cdot y)\cdot c = \varphi(a\cdot x)\cdot(\varphi(y)\cdot c),$$

for all $x, y \in Q$. So

$$\varphi(a)\cdot[\varphi(x)\cdot(\varphi(y)\cdot c)] = \varphi(a\cdot x)\cdot(\varphi(y)\cdot c), \tag{23}$$

for all $x, y \in Q$. Taking $y = \varphi^{-1}\left({}^{-1}c\right)$ in (23), where ${}^{-1}c\cdot c = e$, e is the unit of the loop $(Q,\cdot)$, we obtain

$$\varphi(a)\cdot\varphi(x) = \varphi(a\cdot x), \tag{24}$$

for all $x \in Q$. Now, using (24), from (23) follows:

$$\varphi(a)\cdot[\varphi(x)\cdot(\varphi(y)\cdot c)] = (\varphi(a)\cdot\varphi(x))\cdot(\varphi(y)\cdot c), \tag{25}$$

for all $x, y \in Q$. Denoting $\varphi(x) = u$ and $\varphi(y)\cdot c = v$, (25) implies $\varphi(a)\cdot uv = (\varphi(a)\cdot u)v$, for all $u, v \in Q$, so $\varphi(a) \in N_l^{(\cdot)}$, $\forall a \in N_l^{(\cdot)}$, i.e.

$$\varphi\left(N_l^{(\cdot)}\right) \subseteq N_l^{(\cdot)}. \tag{26}$$

So as the set of all right pseudoautomorphisms is a group, we have that φ^{-1} is a right pseudoautomorphism of the loop $(Q,\cdot)$ as well, so $\varphi^{-1}\left(N_l^{(\cdot)}\right) \subseteq N_l^{(\cdot)}$. Hence, for $a \in N_l^{(\cdot)}$ we get that $\varphi^{-1}(a) \in N_l^{(\cdot)}$, so $a \in \varphi\left(N_l^{(\cdot)}\right)$, i.e

$$N_l^{(\cdot)} \subseteq \varphi\left(N_l^{(\cdot)}\right). \tag{27}$$

From (26) and (27) follows $\varphi\left(N_l^{(\cdot)}\right) = N_l^{(\cdot)}$.

Let $b \in N_m^{(\cdot)}$ and let φ be a *right pseudoautomorphism* of $(Q,\cdot)$ with the companion $c \in Q$. Then, for $\forall x, y \in Q$, the following equalities hold:

$$\varphi(x)\cdot[\varphi(b)\cdot(\varphi(y)\cdot c)] = \varphi(x)\cdot(\varphi(b\cdot y)\cdot c) =$$
$$\varphi(x\cdot by)\cdot c = \varphi(xb\cdot y)\cdot c = \varphi(x\cdot b)\cdot(\varphi(y)\cdot c),$$
$$\varphi(x)\cdot[\varphi(b)\cdot(\varphi(y)\cdot c)] = \varphi(x\cdot b)\cdot(\varphi(y)\cdot c), \tag{28}$$

Taking $y = \varphi^{-1}({}^{-1}c)$ in (28), we have

$$\varphi(x)\cdot\varphi(b) = \varphi(x\cdot b), \tag{29}$$

$\forall x \in Q$. Now, using (29), the equality (28) implies

$$\varphi(x)\cdot[\varphi(b)\cdot(\varphi(y)\cdot c)] = (\varphi(x)\cdot\varphi(b))\cdot(\varphi(y)\cdot c), \tag{30}$$

$\forall x, y \in Q$. Denoting $\varphi(x) = u$, $\varphi(y)\cdot c = v$ in (30) we obtain $u\cdot(\varphi(b)\cdot v) = (u\cdot\varphi(b))v$ for all $u, v \in Q$, i.e. $\varphi(b) \in N_m^{(\cdot)}$, $\forall b \in N_m^{(\cdot)}$, hence

$$\varphi\left(N_m^{(\cdot)}\right) \subseteq N_m^{(\cdot)}. \tag{31}$$

So as φ^{-1} is also a right pseudoautomorphism of the loop $(Q,\cdot)$, we get $\varphi^{-1}\left(N_m^{(\cdot)}\right) \subseteq N_m^{(\cdot)}$. Now, if $b \in N_m^{(\cdot)}$ then $\varphi^{-1}(b) \in N_m^{(\cdot)}$, $\Rightarrow b \in \varphi\left(N_m^{(\cdot)}\right)$, i.e

$$N_m^{(\cdot)} \subseteq \varphi\left(N_m^{(\cdot)}\right). \tag{32}$$

From (31) and (32) follows $\varphi\left(N_m^{(\cdot)}\right) = N_m^{(\cdot)}$.

2. Let $a \subset N_r^{(\cdot)}$ and let φ be a left pseudoautomorphism of $(Q,\cdot)$ with the companion $c \in Q$. Then, for every $x, y \in Q$, the following equalities hold:

$$[(c\cdot\varphi(x))\cdot\varphi(y)]\cdot\varphi(a) = (c\cdot\varphi(x\cdot y))\cdot\varphi(a) = c\cdot\varphi(xy\cdot a) = c\cdot\varphi(x\cdot ya) =$$
$$(c\cdot\varphi(x))\cdot\varphi(ya), \text{ so}$$

$$[(c\cdot\varphi(x))\cdot\varphi(y)]\cdot\varphi(a) = (c\cdot\varphi(x))\cdot\varphi(ya), \tag{33}$$

Taking $x = \varphi^{-1}(c^{-1})$ in (33), where $c\cdot c^{-1} = e$, e is the unit of the loop $(Q,\cdot)$, we obtain

$$\varphi(y)\cdot\varphi(a) = \varphi(y\cdot a), \tag{34}$$

$\forall y \in Q$. Now, using (34), from (33) we get:

$$[(c\cdot\varphi(x))\cdot\varphi(y)]\cdot\varphi(a) = (c\cdot\varphi(x))\cdot(\varphi(y)\cdot\varphi(a)), \tag{35}$$

$\forall x, y \in Q$. Denoting $c \cdot \varphi(x) = u$ and $\varphi(y) = v$, from (35) follows: $uv \cdot \varphi(a) = u \cdot (v \cdot \varphi(a))$, for all $u, v \in Q$, so $\varphi(a) \in N_r^{(\cdot)}$, $\forall a \in N_r^{(\cdot)}$, i.e.

$$\varphi\left(N_r^{(\cdot)}\right) \subseteq N_r^{(\cdot)}. \tag{36}$$

So as $\varphi^{-1} \in PS_r^{(\cdot)}$, we have $\varphi^{-1}\left(N_r^{(\cdot)}\right) \subseteq N_r^{(\cdot)}$. So, for every $a \in N_r^{(\cdot)}$, we get $\varphi^{-1}(a) \in N_r^{(\cdot)} \Rightarrow a \in \varphi\left(N_r^{(\cdot)}\right)$, i.e

$$N_r^{(\cdot)} \subseteq \varphi\left(N_r^{(\cdot)}\right). \tag{37}$$

From (36) and (37) follows $\varphi\left(N_r^{(\cdot)}\right) = N_r^{(\cdot)}$.

Let $b \in N_m^{(\cdot)}$ and let φ be a left pseudoautomorphism of $(Q, \cdot)$ with the companion $c \in Q$. Then, for $\forall x, y \in Q$, the following equalities hold:

$[(c \cdot \varphi(x)) \cdot \varphi(b)] \cdot \varphi(y) = (c \cdot \varphi(x \cdot b)) \cdot \varphi(y) = c \cdot \varphi(xb \cdot y) = c \cdot \varphi(x \cdot by) = (c \cdot \varphi(x)) \cdot \varphi(b \cdot y)$, so

$$[(c \cdot \varphi(x)) \cdot \varphi(b)] \cdot \varphi(y) = (c \cdot \varphi(x)) \cdot \varphi(b \cdot y), \tag{38}$$

$\forall y \in Q$. Taking $x = \varphi^{-1}(c^{-1})$ in (38), we obtain

$$\varphi(b) \cdot \varphi(y) = \varphi(b \cdot y), \tag{39}$$

$\forall x, y \in Q$, and using (39), from (38) follows:

$$[(c \cdot \varphi(x)) \cdot \varphi(b)] \cdot \varphi(y) = (c \cdot \varphi(x)) \cdot (\varphi(b) \cdot \varphi(y)), \tag{40}$$

$\forall x, y \in Q$. Denoting $(c \cdot \varphi(x)) = u$, $\varphi(y) = v$ in (40), we get $(u \cdot \varphi(b)) \cdot v = u \cdot (\varphi(b) \cdot v)$, for all $u, v \in Q$, i.e. $\varphi(b) \in N_m^{(\cdot)}$, $\forall b \in N_m^{(\cdot)}$, so

$$\varphi\left(N_m^{(\cdot)}\right) \subseteq N_m^{(\cdot)}. \tag{41}$$

Analogously, for φ^{-1} we have $\varphi^{-1}\left(N_m^{(\cdot)}\right) \subseteq N_m^{(\cdot)}$. So, if $b \in N_m^{(\cdot)}$ then $\varphi^{-1}(b) \in N_m^{(\cdot)}$, $\Rightarrow b \in \varphi\left(N_m^{(\cdot)}\right)$, i.e

$$N_m^{(\cdot)} \subseteq \varphi\left(N_m^{(\cdot)}\right). \tag{42}$$

From (41) and (42) follows $\varphi\left(N_m^{(\cdot)}\right) = N_m^{(\cdot)}$. □

Corollary. *Let $(Q, \cdot)$ be an arbitrary loop and let φ be a right (left) pseudoautomorphism of $(Q, \cdot)$. Then the restriction $\varphi/N_l^{(\cdot)} \in AutN_l^{(\cdot)}$ (respectively $\varphi/N_r^{(\cdot)} \in AutN_r^{(\cdot)}$) and $\varphi/N_m^{(\cdot)} \in AutN_m^{(\cdot)}$.*

Proof. If φ is a right pseudoautomorphism of $(Q, \cdot)$ then $\varphi\left(N_l^{(\cdot)}\right) = N_l^{(\cdot)}$ and $\varphi\left(N_m^{(\cdot)}\right) = N_m^{(\cdot)}$, so $\varphi/N_l^{(\cdot)}: N_l^{(\cdot)} \to N_l^{(\cdot)}$ and $\varphi/N_m^{(\cdot)}: N_m^{(\cdot)} \to N_m^{(\cdot)}$ are bijections. Moreover, so as the nuclei of the loop $(Q, \cdot)$ are associative subloops, $\varphi/N_r^{(\cdot)} \in AutN_r^{(\cdot)}$ and $\varphi/N_m^{(\cdot)} \in AutN_m^{(\cdot)}$. Analogously, if φ is a left pseudoautomorphism then $\varphi/N_l^{(\cdot)} \in AutN_l^{(\cdot)}$. □

Proposition 6. *Let $(Q, \cdot)$ be a right Bol loop and $c \in Q$. A mapping $\varphi \in S_Q$ is a middle pseudoautomorphism of the loop $(Q, \cdot)$, with the companion c, if and only if φ is a right pseudoautomorphism of $(Q, \cdot)$, with the companion c.*

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Proof. So as $(Q,\cdot)$ is a right Bol loop, the equality $(xc\cdot z)\cdot c = x\cdot(cz\cdot c)$ holds, for every $x,y\in Q$. Taking $c\backslash y = z$ in the last equality, we get $[(x\cdot c)\cdot(c\backslash y)]\cdot c = x\cdot\big[\big(c\cdot(c\backslash y)\big)\cdot c\big] = x\cdot yc$, wich implies $(x\cdot c)\cdot(c\backslash y) = (x\cdot yc)\cdot c^{-1}$, hence

$$(x/c^{-1})\cdot(c\backslash y) = (x\cdot yc)\cdot c^{-1}, \tag{43}$$

for every $x,y\in Q$. If φ is a right pseudoautomorphism of $(Q,\cdot)$, with the companion c, then

$$\varphi(x\cdot y)\cdot c = \varphi(x)\cdot(\varphi(y)\cdot c) \Rightarrow \varphi(x\cdot y) = \big(\varphi(x)\cdot(\varphi(y)\cdot c)\big)\cdot c^{-1},$$

for every $x,y\in Q$, so using (43), we get $\varphi(x\cdot y) = (\varphi(x)/c^{-1})\cdot\big(c\backslash\varphi(y)\big)$, $\forall x,y\in Q$, i.e. φ is a middle pseudoautomorphism of $(Q,\cdot)$, with the companion c. Conversely, if φ is a middle pseudoautomorphism of $(Q,\cdot)$, with the companion c, then $\varphi(x\cdot y) = (\varphi(x)/c^{-1})\cdot\big(c\backslash\varphi(y)\big)$, $\forall x,y\in Q$, so using (43), we get $\varphi(x\cdot y) = \big(\varphi(x)\cdot(\varphi(y)\cdot c)\big)\cdot c^{-1}$, which implies $\varphi(x\cdot y)\cdot c = \varphi(x)\cdot(\varphi(y)\cdot c)$, $\forall x,y\in Q$, i.e. φ is a right pseudoautomorphism of $(Q,\cdot)$, with the companion c. □

Proposition 7. *Let $(Q,\cdot)$ be a left Bol loop and $c\in Q$. A mapping $\varphi\in S_Q$ is a middle pseudoautomorphism of the loop $(Q,\cdot)$, with the companion c if and only if φ is a left pseudoautomorphism of the loop $(Q,\cdot)$, with the companion c^{-1}.*

Proof. Let $(Q,\cdot)$ be a left Bol loop, then $c^{-1}\cdot x = c\backslash x$, for all $x\in Q$, so $c^{-1}(z\cdot c^{-1}x) = (c^{-1}\cdot zc^{-1})x \Leftrightarrow z\cdot c^{-1}x = c\cdot[(c^{-1}\cdot zc^{-1})x]$, for every $x,y\in Q$. Taking $z\to y/c^{-1}$ in the last equality we get

$$(y/c^{-1})\cdot(c^{-1}\cdot x) = c\cdot\Big[\Big(c^{-1}\cdot\big((y/c^{-1})\cdot c^{-1}\big)\Big)\cdot x\Big] \Leftrightarrow$$
$$(y/c^{-1})\cdot(c\backslash x) = c(c^{-1}y\cdot x),$$

for all $x,y\in Q$, where $\varphi\subset S_Q$. So, a mapping $\varphi\subset S_Q$ is a middle pseudoautomorphism of the loop $(Q,\cdot)$, with the companion c, if and only if

$$\varphi(y\cdot x) = (\varphi(y)/c^{-1})\cdot\big(c\backslash\varphi(x)\big) = c\cdot\big[\big(c^{-1}\cdot\varphi(y)\big)\cdot\varphi(x)\big],$$

for $\forall x,y\in Q$, which is equivalent to $c^{-1}\cdot\varphi(y\cdot x) = \big(c^{-1}\cdot\varphi(y)\big)\cdot\varphi(x)$, for all $x,y\in Q$, i.e. if and only if φ is a left pseudoautomorphism with the companion c^{-1}. □

Proposition 8 [10]. *Let $(Q,\cdot)$ be a right Bol loop and let $(Q,\circ)$ be the corresponding middle Bol loop. The following statements hold:*

1. $PS_m^{(\cdot)} = PS_r^{(\cdot)} = PS_r^{(\circ)}$,
2. $PS_l^{(\cdot)} = PS_m^{(\circ)}$,
3. $\alpha\in PS_l^{(\circ)} \Leftrightarrow I\alpha I\in PS_m^{(\cdot)}$, *where* $I: Q\to Q$, $I(x) = x^{-1}$, $\forall x\in Q$.

Proposition 9. *Let $(Q,\cdot)$ be a left Bol loop and let $(Q,\circ)$ be the corresponding middle Bol loop. The following statements hold:*

1. $PS_m^{(\cdot)} = PS_l^{(\cdot)} = PS_r^{(\circ)}$,
2. $PS_r^{(\cdot)} = PS_m^{(\circ)}$,
3. $\alpha\in PS_l^{(\circ)} \Leftrightarrow I\alpha I\in PS_l^{(\cdot)}$.

Proof. 1. Let φ be a middle pseudoautomorphism of $(Q,\cdot)$ with the companion c. Then φ satisfies (5). Using (4), the equality (5) implies:

$\varphi(x//y^{-1}) = [\varphi(x)\circ c]//[(\varphi(y)\backslash\backslash c)^{-1}]^{-1}$

which is equivalent to

$$\varphi(x)\circ c=\varphi(x//y^{-1})\circ[(\varphi(y)\backslash\backslash c)^{-1}]^{-1}, \tag{44}$$

where $(//)$ is the left division in $(Q,\circ)$. So as $(Q,\circ)$ is a middle Bol loop $[(\varphi(y)\backslash\backslash c)^{-1}]^{-1}=\varphi(y)\backslash\backslash c$, $\forall y\in Q$, where "$\backslash\backslash$" is the right division in $(Q,\circ)$, hence (44) is equivalent to

$\varphi(x)\circ c=\varphi(x//y^{-1})\circ(\varphi(y)\backslash\backslash c)$

Denoting $x//y^{-1}$ by z, the last equality implies:

$$\varphi(z\circ y^{-1})\circ c=\varphi(z)\circ(\varphi(y)\backslash\backslash c). \tag{45}$$

For $z=y=e$, from (45) follows $\varphi(e)=e$. Taking $z=e$ in (45) and using the equality $\varphi(e)=e$, we get $\varphi(y^{-1})\circ c=\varphi(y)\backslash\backslash c$, so (45) is equivalent to

$$\varphi(z\circ y^{-1})\circ c=\varphi(z)\circ\varphi(y^{-1})\circ c,$$

i.e. φ is a right pseudoautomorphism $(Q,\circ)$ with the companion c. Conversely, if $\varphi\in PS_r^{(\circ)}$ then $\exists c\in Q$: $\varphi(x\circ y)\circ c=\varphi(x)\circ(\varphi(y)\circ c)$, $\forall x,y\in Q$. Using (3), from the last equality we get

$$\varphi(x/y^{-1})/c^{-1}=\varphi(x)/(\varphi(y)/c^{-1})^{-1}\Leftrightarrow\varphi(x)=(\varphi(x/y^{-1})/c^{-1})\cdot I(\varphi(y)/c^{-1}).$$

Denoting $x/y^{-1}=z$, the previous equality implies

$$\varphi(z\cdot y^{-1})=(\varphi(z)/c^{-1})\cdot I(\varphi(y)/c^{-1}). \tag{46}$$

Taking $z=e$, from (46) follows $\varphi(y^{-1})=c\cdot I(\varphi(y)/c^{-1})$, hence $c^{-1}\cdot\varphi(y^{-1})=I(\varphi(y)/c^{-1})$, so (46) implies $\varphi(z\cdot y^{-1})=(\varphi(z)/c^{-1})\cdot(c^{-1}\cdot\varphi(y^{-1}))$, which is equivalent to $\varphi(z\cdot y^{-1})=(\varphi(z)/c^{-1})\cdot(c\backslash\varphi(y^{-1}))$, $\forall y,z\in Q$, i.e. φ is a middle pseudoautomorphism of $(Q,\cdot)$, with the companion c, and $PS_m^{(\cdot)}=PS_l^{(\cdot)}=PS_r^{(\circ)}$.

2. Let φ be a right pseudoautomorphism of the loop $(Q,\cdot)$, with the companion b:

$$\varphi(x\cdot y)\cdot b=\varphi(x)\cdot(\varphi(y)\cdot b), \tag{47}$$

$\forall x,y\in Q$. Using (4) in (47), we get

$$\varphi(x//y^{-1})//b^{-1}=\varphi(x)//(\varphi(y)//b^{-1})^{-1}. \tag{48}$$

So as $(Q,\circ)$ is a middle Bol loop, denoting $(\varphi(y)//b^{-1})^{-1}=u$, we have: $\varphi(y)//b^{-1}=u^{-1}\Leftrightarrow\varphi(y)=u^{-1}\circ b^{-1}\Leftrightarrow b\circ u=(\varphi(y))^{-1}\Leftrightarrow u=b\backslash\backslash(\varphi(y))^{-1}$, so

$$(\varphi(y)//b^{-1})^{-1}=b\backslash\backslash(\varphi(y))^{-1}. \tag{49}$$

Using (49), the equality (48) implies

$$\varphi(x//y^{-1})//b^{-1}=\varphi(x)//\left(b\backslash\backslash(\varphi(y))^{-1}\right). \tag{50}$$

Denoting $x//y^{-1}=z$, i.e. $z\circ y^{-1}=x$, (50) implies

$$\varphi(z)//b^{-1}=\varphi(z\circ y^{-1})//\left(b\backslash\backslash(\varphi(y))^{-1}\right),$$

which is equivalent to:

$$\varphi(z\circ y^{-1})=(\varphi(z)//b^{-1})\circ\left(b\backslash\backslash(\varphi(y))^{-1}\right). \tag{51}$$

Taking $z=e$ in (51), we get

$$\varphi(y^{-1})=b\circ\left(b\backslash\backslash(\varphi(y))^{-1}\right)=(\varphi(y))^{-1},$$

$\forall y\in Q$, so (51) is equivalent to

$$\varphi(z\circ y^{-1})=(\varphi(z)//b^{-1})\circ(b\backslash\backslash\varphi(y^{-1})),$$

$\forall y,z\in Q$, i.e. $\varphi\in PS_m^{(\circ)}$, with the companion b. Conversely, if $\varphi\in PS_m^{(\circ)}$ then $\exists c\in Q$:

$$\varphi(x\circ y)=(\varphi(x)//c^{-1})\circ(c\backslash\backslash\varphi(y)), \tag{52}$$

$\forall x,y\in Q$. Denoting $\varphi(x)//c^{-1}=u$ and using (3), we get $u\circ c^{-1}=\varphi(x)$, $u/c=\varphi(x)$, $\varphi(x)\cdot c=u$, so

$$\varphi(x)//c^{-1}=\varphi(x)\cdot c. \tag{53}$$

Analogously, denoting $c\backslash\backslash\varphi(y)=v$ and using (3), we get: $c\circ v=\varphi(y)$, $c/v^{-1}=\varphi(y)$, $\varphi(y)\cdot v^{-1}=c$, $v^{-1}=\varphi(y)^{-1}\cdot c$, $v=I(I\varphi(y)\cdot c)$, so

$$c\backslash\backslash\varphi(y)=I(I\varphi(y)\cdot c), \tag{54}$$

$\forall y\in Q$. Now, using (53) and (54), the equality (52) implies

$$\varphi(x/y^{-1})=(\varphi(x)\cdot c)/((I\varphi(y)\cdot c)),$$

which is equivalent to

$$\varphi(x)\cdot c=\varphi(x/y^{-1})\cdot(I\varphi(y)\cdot c).$$

Denoting $x/y^{-1}=z$ in the last equality, we get

$$\varphi(z\cdot y^{-1})\cdot c=\varphi(z)\cdot(I\varphi(y)\cdot c). \tag{55}$$

Taking $z=e$, from (55) follows $\varphi(y^{-1})\cdot c=I\varphi(y)\cdot c,\Rightarrow\varphi(y^{-1})=I\varphi(y)$, $\forall y\in Q$, so

$$\varphi I=I\varphi. \tag{56}$$

From (55) and (56), we get

$$\varphi(z\cdot y^{-1})\cdot c=\varphi(z)\cdot(\varphi(y^{-1})\cdot c),$$

$\forall y,z\in Q$, i.e. $\varphi\in PS_m^{(\cdot)}$. So $PS_r^{(\cdot)}=PS_m^{(\circ)}$.

3. If $\varphi\in PS_l^{(\circ)}$ then there exists an element $c\in Q$, such that, for $\forall x,y\in Q$,

$$c\circ\varphi(x\circ y)=(c\circ\varphi(x))\circ\varphi(y). \tag{57}$$

Using (3), the equality (57) takes the form $c/\varphi(x/y^{-1})^{-1}=(c/\varphi(x)^{-1})/\varphi(y)^{-1}$ $\Leftrightarrow$ $c/\varphi(x)^{-1}=[c/\varphi(x/y^{-1})^{-1}]\cdot\varphi(y)^{-1}$, so denoting $x/y^{-1}=z$, we get

$$c/I\varphi(z\cdot y^{-1})=[c/I\varphi(z)]\cdot I\varphi(y), \tag{58}$$

which (for $z=e$) implies $c/I\varphi I(y)=c\cdot I\varphi(y)$, so $c/I\varphi(y)=c\cdot I\varphi I(y)$, $\forall x,y\in Q$, i.e.

$$c/I\varphi=c\cdot I\varphi I. \tag{59}$$

Using (59), from (58) follows $c\cdot I\varphi I(z\cdot y)=[c\cdot I\varphi I(z)]\cdot I\varphi I(y)$, $\forall y,z\in Q$, i.e. $I\varphi I$ is a left pseudoautomorphism of $(Q,\cdot)$, with the companion c. □

Corollary. *Let $(Q,\circ)$ be a middle Bol loop and let φ be a middle pseudoautomorphism of $(Q,\circ)$, then $\varphi\left(N_l^{(\circ)}\right)=N_l^{(\circ)}$ and $\varphi\left(N_r^{(\circ)}\right)=N_r^{(\circ)}$.*

Proof. Let $(Q,\cdot)$ be the corresponding right Bol loop of $(Q,\circ)$, then $PS_l^{(\cdot)}=PS_m^{(\circ)}$, so φ is a left pseudoautomophism of $(Q,\cdot)$. Also the equality $N_r^{(\cdot)}=N_r^{(\circ)}$ holds, so $\varphi\left(N_r^{(\cdot)}\right)=N_r^{(\cdot)}=N_r^{(\circ)}=N_l^{(\circ)}$. □

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