STUDIA UNIVERSITATIS MOLDAVIAE, 2016, nr.2(92)
Seria "Științe exacte și economice" ISSN 1857-2073 ISSN online 2345-1033 p.40-45

ON SOME IDENTITIES IN TERNARY QUASIGROUPS

Dina CEBAN

State University of Moldova

Identities of length 5, with two variables in binary quasigroups are called minimal identities. V.Belousov and, independently, F. Bennett showed that, up to the parastrophic equivalence, there are seven minimal identities. The existence of paratopies of orthogonal systems, consisting of two binary quasigroups and the binary selectors, implies three minimal identities (of seven). The existence of paratopies of orthogonal system, consisting of three ternary quasigroups and the ternary selectors, gives 67 identities. In the present article these identities are listed and it is proved that each of 67 identities is equivalent to one of the following four identities: ${}^{\alpha}A({}^{\beta}A,\ {}^{\gamma}A,\ {}^{\delta}A) = E_1$, ${}^{\alpha}A({}^{\beta}A,\ {}^{\gamma}A,\ E_1) = E_2$, ${}^{\alpha}A({}^{\beta}A,\ E_1,\ E_2) = {}^{\gamma}A({}^{\delta}A,\ E_1,\ E_3)$, ${}^{\alpha}A({}^{\beta}A,\ E_1,\ E_2) = {}^{\gamma}A({}^{\delta}A,\ E_1,\ E_2)$, where A is a ternary quasigroup and $\alpha,\beta,\gamma,\delta \in S_4$. A necessary condition when a tuple $\theta = (A_1, A_2, \dots, A_n)$ consisting of n-ary quasigroups, defined on a set Q, is a paratopy of the orthogonal system $\Sigma = \{A_1, A_2, \dots, A_n, E_1, E_2, \dots, E_n\}$ is given.

Keywords: *minimal identity, n-ary quasigroup, paratopy, orthogonal system of quasigroups.*

ASUPRA UNOR IDENTITĂȚI ÎN CVASIGRUPURI TERNARE

Identități minimale în cvasigrupuri binare se numesc identitățile de lungime 5, cu două variabile. V.Belousov şi, independent, F.Bennett au demonstrat că, abstracție facând de relația de echivalență parastrofică, există şapte identități minimale. Paratopiile sistemelor ortogonale formate din două cvasigrupuri binare şi cei doi selectori binari implică apariția a trei identități minimale (din şapte). În caz ternar, paratopiile sistemelor ortogonale din trei cvasigrupuri ternare şi cei trei selectori ternari conduc la apariția a 67 de identități. În articol este prezentată lista acestor identități şi se demonstrează că oricare dintre aceste 67 de identități este echivalentă cu una din următoarele patru identități: ${}^{\alpha}A({}^{\beta}A,\ {}^{\gamma}A,\ {}^{\delta}A) = E_1$, ${}^{\alpha}A({}^{\beta}A,\ {}^{\gamma}A,\ E_1) = E_2$, ${}^{\alpha}A({}^{\beta}A,\ E_1,\ E_2) = {}^{\gamma}A({}^{\delta}A,\ E_1,\ E_3)$, ${}^{\alpha}A({}^{\beta}A,\ E_1,\ E_2) = {}^{\gamma}A({}^{\delta}A,\ E_1,\ E_2)$, unde A este un cvasigrup ternar şi $\alpha,\beta,\gamma,\delta \in S_4$. De asemenea, este dată o condiție necesară ca o uplă $\theta = (A_1, A_2, \dots, A_n)$ formată din cvasigrupuri n-are, definite pe o mulțime Q, să fie o paratopie a sistemului ortogonal $\Sigma = \{A_1, A_2, \dots, A_n, E_1,\ E_2, \dots, E_n\}$.

Cuvinte-cheie: *identitate minimală, cvasigrup n-ar, paratopie, sistem ortogonal de cvasigrupuri.*

Let Q be a nonempty set and let n be a positive integer. A n-groupoid (Q, A) is called a n-quasigroup if in the equality $A(x_1, x_2, \dots, x_n) = x_{n+1}$ any element of the set $\{x_1, x_2, \dots, x_{n+1}\}$ is uniquely determined by the remaining n elements. If (Q, A) is an n-ary quasigroup and $\sigma \in S_n$, then the operation ${}^{\sigma}A$ defined by the equivalence: ${}^{\sigma}A(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = x_{\sigma(n+1)} \Leftrightarrow A(x_1, x_2, \dots, x_n) = x_{n+1}$, for every $x_1, x_2, \dots, x_n \in Q$, is called a σ-parastrophe (or, simply, a parastrophe) of (Q, A). We will denote the transposition $(i, n+1)$, where $i \in \{1, 2, \dots, n\}$, by π_i, so ${}^{(i,n+1)}A = {}^{\pi_i}A$. A σ-parastrophe of an n-ary quasigroup (Q, A) is called a principal parastrophe if $\sigma(n+1) = n+1$. The n-ary operations $A_1, A_2, \dots, A_n$, defined on Q, are called orthogonal if the system of equations $\{A_i(x_1, x_2, \dots, x_n) = a_i,\ i = \overline{1, n}$ has a unique solution, for every $a_1, a_2, \dots, a_n \in Q$ [1]. A system of n-ary operations $A_1, A_2, \dots, A_s$, defined on a set Q, where $s \geq n$, is called orthogonal if every n operations of this system are orthogonal. n-Ary quasigroups, for which there exist n orthogonal parastrophes (principal parastrophes) are called parastrophic-orthogonal (self-orthogonal). The operations $E_1, E_2, \dots, E_n$, where $E_i(x_1, x_2, \dots, x_n) = x_i$, for every $x_1, x_2, \dots, x_n \in Q$, are called n-ary selectors. If $\Sigma = \{A_1, A_2, \dots, A_n, E_1, E_2, \dots, E_n\}$, where $A_1, A_2, \dots, A_n$ are n-quasigroups, is an orthogonal system then we denote $\{A_1\theta, A_2\theta, \dots, A_n\theta, E_1\theta, E_2\theta, \dots, E_n\theta\}$ by $\Sigma\theta$. A bijection $\theta: Q^n \to Q^n$ is called a paratopy of the system Σ if $\Sigma\theta = \Sigma$. V. Belousov proved in [2] that there exist nine orthogonal systems, consisting of two binary quasigroup operations and two binary selectors, having at least one non trivial paratopy. V. Belousov and, independently, F. Bennett showed that, up to the parastrophic equivalence, there are seven minimal identities [3]. The following minimal identities from Belousov-Bennett classification are implied by the existence of paratopies: $x \cdot (x \cdot xy) = y$ (the identity of type T_1); $xy \cdot x = y \cdot xy$ (the second Stein law, type T_6); $yx \cdot xy = x$ (the third Stein law, type T_{10}). The minimal identity of type T_1 involves the orthogonality of the parastrophes: $\varepsilon \perp r, rl \perp s$; the minimal identity of type T_6 involves the orthogonality of the parastrophes: $\varepsilon \perp lr$, $\varepsilon \perp rl$, $r \perp lr$, $l \perp rl$, $r \perp s$ and $l \perp s$; the minimal identity of type

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STUDIA UNIVERSITATIS MOLDAVIAE, 2016, nr.2(92)
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T_{10} involves the orthogonality of the parastrophes: $\varepsilon \perp s$, $\varepsilon \perp r$, $rl \perp lr$, where $s = (12), r = (23)$ and $l = (13)$ [4, 5].

We consider below a necessary condition when a tuple $\theta = (A_1, A_2, \ldots, A_n)$ consisting of n-ary quasigroups, defined on a set Q, is a paratopy of the orthogonal system $\Sigma = \{A_1, A_2, \ldots, A_n, E_1, E_2, \ldots, E_n\}$.

Theorem 1. Let $\Sigma = \{A_1, A_2, \ldots, A_n, E_1, E_2, \ldots, E_n\}$ and $\theta = (A_1, A_2, \ldots, A_n)$, where $A_1, A_2, \ldots, A_n$ are n-ary quasigroups, defined on a set Q and $E_1, E_2, \ldots, E_n$ are n-ary selectors on Q, then system of equalities $A_1\theta = E_2, A_2\theta = E_3, \ldots, A_n\theta = E_1$ is equivalent to the system of conditions $A_2 = {}^{\alpha^{-1}}A_1, A_3 = {}^{\alpha^{-2}}A_1, \ldots, A_n = {}^{\alpha^{-n+1}}A_1$ and (Q, A_1) satisfies the identity $A_1(A_1, {}^{\alpha^{-1}}A_1, {}^{\alpha^{-2}}A_1, \ldots, {}^{\alpha^{-n+1}}A_1) = E_2$.

Proof. Let $\Sigma = \{A_1, A_2, \ldots, A_n, E_1, E_2, \ldots, E_n\}$ be an orthogonal system and let $\theta = (A_1, A_2, \ldots, A_n)$, where $A_1, A_2, \ldots, A_n$ are n-ary quasigroups, defined on a set Q and $E_1, E_2, \ldots, E_n$ are n-ary selectors on Q. If $A_1\theta = E_2, A_2\theta = E_3, \ldots, A_{n-1}\theta = E_n, A_n\theta = E_1$, then $\theta^2 = (E_2, \ldots, E_n, E_1)$, $\theta^3 = (A_2, \ldots, A_n, A_1)$, $\theta^4 = (E_3, \ldots, E_n, E_1, E_2)$, ..., $\theta^{2n-1} = (A_n, A_1, \ldots, A_{n-1})$, $\theta^{2n} = (E_1, E_2, \ldots, E_n)$, so the order of the mapping θ is $2n$. From $A_1\theta = E_2$ it follows $A_1\theta^2 = A_2$, i.e. $A_2 = A_1(E_2, \ldots, E_n, E_1)$, so $A_2 = {}^{\alpha^{-1}}A_1$, where $\alpha = (12\ldots n)$. Also, $A_1\theta = E_2$ implies $A_1\theta^4 = E_2\theta^3 = A_3$, i.e. $A_3 = A_1(E_3, \ldots, E_n, E_1, E_2)$, hence $A_3 = {}^{\alpha^{-2}}A_1$. Analogously, we obtain, for every $i = 2, n$,

$$A_i = {}^{\alpha^{-i+1}}A_1. \tag{1}$$

Using the equalities (1) in $A_1\theta = E_2$, we get

$$A_1(A_1, {}^{\alpha^{-1}}A_1, {}^{\alpha^{-2}}A_1, \ldots, {}^{\alpha^{-n+1}}A_1) = E_2. \tag{2}$$

Conversely, if (2) and (1) hold, for every $i = 2, n$, then

$$A_1(A_1, A_2, \ldots, A_n) = E_2, \tag{3}$$

so $A_1\theta = E_2$. Moreover, for all $i = 2, n$, using the parastrophy, the equality (3) implies

$${}^{\alpha^{-i+1}}A_1(A_{\alpha^{-i+1}(1)}, A_{\alpha^{-i+1}(2)}, \ldots, A_{\alpha^{-i+1}(n)}) = E_2. \tag{4}$$

Now, according to (1) we have: $A_{\alpha^{-i+1}(k)} = A_{i+k-1} = {}^{\alpha^{-i-k+2}}A_1 = {}^{\alpha^{-i+1}}({}^{\alpha^{-k+1}}A_1) = {}^{\alpha^{-i+1}}A_k, \forall k = 1, n$. From the equalities $A_{\alpha^{-i+1}(k)} = {}^{\alpha^{-i+1}}A_k, \forall k = 1, n$, and (4), we get $A_i({}^{\alpha^{-i+1}}A_1, {}^{\alpha^{-i+1}}A_2, \ldots, {}^{\alpha^{-i+1}}A_n) = E_2$. Now, using the parastrophic transformation in the last equality, we have

$$A_i(A_1(x_{\alpha^{i-1}(1)}^{\alpha^{i-1}(n)}), A_2(x_{\alpha^{i-1}(1)}^{\alpha^{i-1}(n)}), \ldots, A_n(x_{\alpha^{i-1}(1)}^{\alpha^{i-1}(n)})) = x_2,$$

hence, denoting $x_{\alpha^{i-1}(k)}$ by $x_k, \forall k = 1, n$, we obtain $A_i(A_1(x_1^n), A_2(x_1^n), \ldots, A_n(x_1^n)) = x_{i+1}$, which implies $A_i\theta = E_{i+1}$, $i = 2, n$. □

Remark. It is known that n-ary quasigroups satisfying the identity (2) are self-orthogonal (see, for example, [6, 7]).

Orthogonal systems Σ, consisting of three ternary quasigroups and three ternary selectors, are partly considered in [8], where are found all paratopies of such systems as triples which components are three ternary quasigroup operations or two ternary quasigroup opeartions and one ternary selector. In the second part of this investigation we consider all paratopies (of such systems) as triples which components are two ternary selectors and a ternary quasigroup operation or three ternary selectors. We prove that there exist 153 orthogonal systems consisting of three ternary quasigroup operations and three ternary selectors, which admit at least one non trivial paratopy. Let $\Sigma = A_1, A_2, A_3, E_1, E_2, E_3$ be an ortogonal system, where A_1, A_2, A_3 are ternary quasigroup operations, defined on a set Q, and E_1, E_2, E_3 are the ternary selectors on Q. We show [9,10] that the existence of nontrivial paratopies of Σ implies the folowing 67 identities, where $A \in \{A_1, A_2, A_3\}$:

1. $A(A, {}^{(132)}A, {}^{(123)}A) = E_2$;
2. $A(A, {}^{(132)}A, {}^{(123)}A) = E_3$;
3. $A\ E_1, A, {}^{23}\ A\ = E_3$;
4. $A(E_1, E_3, {}^{\pi_2}A(E_1, E_2, A)) = {}^{\pi_3}A(E_1, A, E_3)$;
5. $A(E_1, {}^{\pi_3}A(E_1, A, E_3), E_2) = {}^{\pi_2}A(E_1, E_2, A)$;
6. $A({}^{\pi_2}A(E_3, A, E_2), E_1, {}^{\pi_1}A(A, E_3, E_1)) = A$;
7. $A({}^{\pi_2}A(E_2, E_1, A), {}^{\pi_3}A(A, E_1, E_3), E_1) = E_2$;
8. $A({}^{\pi_3}A(A, E_3, {}^{\pi_3}A(E_2, A, E_3)), E_1, A) = E_2$;

9. $A({}^{\pi_2}A(E_2,E_3,A),{}^{\pi_1}A(E_3,E_1,A),E_3)=A$;
10. $A({}^{\pi_3}A(E_2,E_3,A),{}^{\pi_1}A(A,E_1,E_2),E_1)=A$;
11. $A(E_3,A,{}^{\pi_2}A(E_2,{}^{\pi_2}A(A,E_2,E_1),A))=E_1$;
12. $A({}^{\pi_3}A(E_3,A,E_2),E_2,{}^{\pi_1}A(E_2,A,E_1))=A$;
13. $A(A,{}^{\pi_1}A(E_3,A,E_1),E_2)={}^{\pi_2}A(E_3,E_2,A)$;
14. $A(E_2,{}^{\pi_1}A(A,E_3,E_1),{}^{\pi_2}A(E_3,A,E_2))=A$;
15. $A({}^{\pi_3}A(E_2,A,E_3),{}^{\pi_1}A(E_2,E_1,A),E_2)=E_1$;
16. $A(E_2,{}^{\pi_3}A(E_3,A,{}^{\pi_3}A(A,E_1,E_3)),A)=E_1$;
17. $A({}^{\pi_2}A(E_2,E_3,A),{}^{\pi_1}A(E_3,E_1,A),E_3)=A$;
18. $A(A,E_2,{}^{(13)}A)=E_3$;
19. $A({}^{\pi_3}A(A,E_2,E_3),E_2,E_1)={}^{\pi_1}A(E_1,E_2,A)$;
20. $A(E_3,E_2,{}^{\pi_1}A(E_1,E_2,A)={}^{\pi_3}A(A,E_2,E_3)$;
21. $A(E_1,{}^{\pi_3}A(A,E_3,E_1),{}^{\pi_2}A(A,E_1,E_2))=A$;
22. $A(A,E_3,{}^{\pi_1}A({}^{\pi_1}A(E_1,A,E_2),E_1,A)))=E_2$;
23. $A({}^{\pi_2}A(E_2,A,E_1),{}^{\pi_3}A(E_3,E_1,A),E_2)=A$;
24. $A({}^{\pi_2}A(A,E_3,E_2),A,E_1)={}^{\pi_1}A(E_1,E_3,A)$;
25. $A(E_3,{}^{\pi_3}A(E_2,E_3,A),{}^{\pi_1}A(A,E_1,E_2))=A$;
26. $A({}^{\pi_2}A(E_3,E_2,A),E_3,{}^{\pi_1}A(E_3,A,E_1))=E_1$;
27. $A(E_2,{}^{\pi_3}A(E_3,A,E_1),A)={}^{\pi_2}A(A,E_2,E_1)$;
28. $A({}^{\pi_3}A(E_3,A,E_2),E_2,{}^{\pi_1}A(E_2,A,E_1))=A$;
29. $A(E_1,{}^{\pi_3}A(A,E_3,E_1),{}^{\pi_2}A(A,E_1,E_2))=A$;
30. $A({}^{\pi_3}A(E_3,E_1,A),E_3,{}^{\pi_2}A(E_2,A,E_1))=A$;
31. $A({}^{\pi_3}A(A,E_3,E_2),E_1,A)={}^{\pi_1}A(E_1,A,E_2)$;
32. $A(E_3,{}^{\pi_1}A(E_1,E_3,A),{}^{\pi_2}A(A,E_3,E_2))=E_2$;
33. $A(A,{}^{(13)}A,E_3)=E_2$;
34. $A({}^{\pi_2}A(A,E_2,E_3),E_1,E_3)={}^{\pi_1}A(E_1,A,E_3)$;
35. $A(E_2,{}^{\pi_1}A(E_1,A,E_3),E_3)={}^{\pi_2}A(A,E_2,E_3)$;
36. $A(E_1,E_2,A(E_1,E_2,A))={}^{\pi_3}A$;
37. $A(E_1,E_2,A(E_2,E_1,A))={}^{(12)\pi_3}A$;
38. $A(E_1,{}^{(23)\pi_3}A,A(E_1,A,E_2))=E_3$;
39. $A(E_1,{}^{\pi_3}A(E_1,E_3,A),{}^{\pi_2}A(E_1,A,E_2))=A$;
40. ${}^{(132)\pi_2}A(E_2,{}^{(132)\pi_2}A,E_1)=A(E_3,E_1,A)$;
41. ${}^{\pi_2}A({}^{\pi_3}A(E_3,E_1,A),A,{}^{\pi_2}A(E_2,A,E_1))=E_3$;
42. $A(A,E_3,A(E_3,E_1,A))={}^{(132)\pi_2}A$;
43. $A(E_3,{}^{(132)\pi_1}A,A(A,E_1,E_2))=E_2$;
44. $A(E_3,{}^{\pi_3}A(E_2,E_3,A),{}^{\pi_1}A(A,E_1,E_2))=A$;
45. $A({}^{(13)\pi_3}A,E_2,A(A,E_2,E_1))=E_3$;
46. $A({}^{\pi_3}A(E_3,E_2,A),E_2,{}^{\pi_1}A(A,E_2,E_1))=A$;
47. $A(E_1,A(E_1,E_3,A),{}^{(23)\pi_2}A)=E_2$;
48. $A(E_1,{}^{\pi_3}A(E_1,E_3,A),{}^{\pi_2}A(E_1,A,E_2))=A$;
49. $A(A,A(E_2,A,E_1),E_2)={}^{(123)\pi_3}A$;
50. $A({}^{\pi_2}A(E_2,A,E_1),{}^{\pi_3}A(E_3,E_1,A),E_2)=A$;
51. $A(E_1,A(E_1,A,E_3),E_3)={}^{\pi_2}A$;
52. $A(E_1,A(E_3,A,E_1),E_3)={}^{(13)\pi_2}A$;
53. $A({}^{\pi_2}A(E_2,A,E_3),{}^{\pi_1}A(A,E_1,E_3),E_3)=A$;
54. $A({}^{(12)\pi_2}A,A(A,E_1,E_3),E_3)=E_2$;
55. $A(E_2,A(A,E_3,E_1),{}^{(123)\pi_2}A)=E_3$;
56. $A(E_2,{}^{\pi_1}A(A,E_3,E_1),{}^{\pi_2}A(E_3,A,E_2))=A$;
57. $A({}^{\pi_3}A(E_2,E_3,A),{}^{\pi_1}A(A_2,E_1,E_2),E_1)=A$;
58. $A(A(E_2,E_3,A),{}^{(123)\pi_1}A,E_1)=E_2$;
59. $A(A(E_3,E_2,A),E_2,{}^{(123)\pi_1}A)=E_1$;
60. $A({}^{\pi_3}A(E_3,E_2,A),E_2,{}^{\pi_1}A(A,E_2,E_1))=A$;

61. $A(A(A,E_2,E_1),E_2,A) ={}^{(123)\pi_3} A$;
62. $A(A(E_2,A,E_3),{}^{(12)\pi_1} A,E_3) = E_1$;
63. $A({}^{\pi_2}A(E_2,A,E_3),{}^{\pi_1} A(A,E_1,E_3),E_3) = A$;
64. $A(A(E_3,A,E_2),E_1,{}^{(132)\pi_1} A) = E_3$;
65. $A({}^{\pi_2}A(E_3,A,E_2),E_1,{}^{\pi_1} A(A,E_3,E_1)) = A$;
66. $A(A(A,E_2,E_3),E_2,E_3) ={}^{\pi_1} A$;
67. $A(A(A,E_3,E_2),E_2,E_3) ={}^{(23)\pi_1} A$;

Theorem 2. *Every of the given above 67 identities on ternary quasigroups is equivalent to one of the following four identities:*

I. ${}^{\alpha}A({}^{\beta}A,\ {}^{\gamma}A,\ {}^{\delta}A) = E_1$,

II. ${}^{\alpha}A({}^{\beta}A,\ {}^{\gamma}A,E_1) = E_2$,

III. ${}^{\alpha}A({}^{\beta}A,E_1,E_2) = {}^{\gamma}A({}^{\delta}A,E_1,E_3)$,

IV. ${}^{\alpha}A({}^{\beta}A,E_1,E_2) = {}^{\gamma}A({}^{\delta}A,E_1,E_2)$,

where A is a ternary quasigroup and $\alpha,\beta,\gamma,\delta \in S_4$.

Proof. 1. $A(A,\ {}^{(132)}A,\ {}^{(123)}A) = E_2 \Leftrightarrow A(\ {}^{12} A(x_2,x_1,x_3),{}^{(23)} A(x_2,x_1,x_3),{}^{(13)}\ A(x_2,x_1,x_3)) = E_1(x_2,x_1,x_3)$, i.e. the identity takes the form I: $A(\ {}^{12} A,{}^{(23)} A,{}^{(13)} A) = E_1$.

2. $(A,{}^{(132)} A,{}^{(123)} A) = E_3 \Leftrightarrow A(\ {}^{13} A(x_3,x_2,x_1),{}^{(12)} A(x_3,x_2,x_1),{}^{(23)} A(x_3,x_2,x_1)) = E_1(x_3,x_2,x_1)$, so we get $A(\ {}^{13} A,{}^{(12)} A,{}^{(23)} A) = E_1$, i.e. an identity of the form I.

3. $A\ \ E_1,A,\ {}^{23}\ A\ \ = E_3 \Leftrightarrow A\ \ E_1\ \ x_1,x_3,x_2\ \ ,\ {}^{23}\ A\ \ x_1,x_3,x_2\ \ ,A(x_1,x_3,x_2)\ \ = E_2(x_1,x_3,x_2)$, which is an identity of the form II: ${}^{(13)}A\ \ A,\ \ {}^{23}\ A,E_1\ \ = E_2$.

18. $A(A,E_2,{}^{(13)} A) = E_3 \Leftrightarrow A(\ {}^{123}\ A(x_2,x_3,x_1),E_1(x_2,x_3,x_1),\ {}^{23}\ A(x_2,x_3,x_1)) = E_2(x_2,x_3,x_1)$, we obtain the identity of the form II: ${}^{(23)}A({}^{(123)}A,\ {}^{(23)}A,E_1) = E_2$.

33. $A(A,{}^{(13)} A,E_3) = E_2 \Leftrightarrow A({}^{(13)}A,A,E_1) = E_2$, so it is reduced to an identity of the form II.

36. $A(E_1,E_2,A(E_1,E_2,A)) ={}^{\pi_3} A \Leftrightarrow {}^{\pi_3}A\ \ E_1,E_2,{}^{\pi_3}A\ \ = A(E_1,E_2,A)$, i.e. the identity is equivalent to one of the form IV: ${}^{\pi_3}A\ \ {}^{\pi_3}A,E_1,E_2\ \ = A(A,E_1,E_2)$.

37. $A(E_1,E_2,A(E_2,E_1,A)) ={}^{(12)\pi_3} A \Leftrightarrow {}^{\pi_3}A\ \ E_1,E_2,\ \ {}^{12\ \pi_3}A\ \ = A(E_2,E_1,A)$, so we get ${}^{(123)\pi_3}A\ \ \ {}^{12\ \pi_3}A,E_1,E_2\ \ = {}^{(23)}A(A,E_1,E_2)$ – an identity of the form IV.

38. $A(E_1,\ {}^{(23)\pi_3}A,A(E_1,A,E_2)) = E_3 \Leftrightarrow {}^{\pi_3}A(E_1,\ \ {}^{23\ \pi_3}A,E_3) = A(E_1,A,E_2)$, hence we obtain the identity of the form III: ${}^{\pi_3}A(\ {}^{23\ \pi_3}A,E_1,E_3) = A(A,E_1,E_2)$.

40. ${}^{(132)\pi_2}A(E_2,\ {}^{(132)\pi_2}A,E_1) = A(E_3,E_1,A) \Leftrightarrow {}^{\pi_2}A(\ {}^{23\ \pi_2}A,E_1,E_2) = A(\ {}^{13}\ A,\ E_1,\ E_3)$, which is an identity of the form III.

43. $A(E_3,\ {}^{(132)\pi_1}A,\ A(A,\ E_1,\ E_2)) = E_2 \Leftrightarrow {}^{\pi_3}A(E_3,\ \ {}^{132\ \pi_1}A,\ E_2) = A(A,\ E_1,\ E_2) \Leftrightarrow {}^{(132)\pi_3}A({}^{(13)\pi_1}A,E_1,E_3) = {}^{(23)}A({}^{(12)}A,E_1,E_2)$ – an identity of the form III.

45. $A(\ {}^{13\ \pi_3}A,E_2,A(A,E_2,E_1)) = E_3 \Leftrightarrow {}^{\pi_3}A(\ {}^{123\ \pi_3}A,E_1,E_3) = A\ \ \ {}^{12}\ A,E_1,E_2$, which is an identity of the form III.

47. $A\ \ E_1,\ A\ \ E_1,\ E_3,\ A\ \ ,\ \ {}^{23\ \pi_2}A\ \ = E_2 \Leftrightarrow {}^{\pi_2}A\ \ E_1,\ E_2,\ \ {}^{23\ \pi_2}A\ \ = A\ \ E_1,\ E_3,\ A \Leftrightarrow {}^{\pi_2}A({}^{(23)\pi_2}A,E_1,E_2) = A(A,E_1,E_3)$, which has the form III.

51. $A(E_1,A(E_1,A,E_3),E_3) ={}^{\pi_2} A \Leftrightarrow {}^{\pi_2}A\ \ E_1,{}^{\pi_2}A\ ,E_3\ \ = A\ \ E_1,A,E_3\ \ \Leftrightarrow {}^{\pi_2}A\ \ \ {}^{23\ \pi_2}A,E_1,E_2\ \ = A\ \ \ {}^{23}\ A,E_1,E_2$ – an identity of the form IV.

52. $A(E_1,\ A(E_3,\ A,\ E_1),\ E_3) = {}^{(13)\pi_2}A \Leftrightarrow {}^{\pi_2}A\ \ E_1\ ,\ {}^{13\ \pi_2}A,\ E_3\ \ = A(E_3,A,E_1) \Leftrightarrow {}^{(12)\pi_2}A\ \ \ {}^{132\ \pi_2}A,E_1,E_2\ \ = {}^{(132)}A\ \ \ {}^{23}\ A,E_1,E_2$, which has the form IV.

54. $(\ {}^{(12)\pi_2}A,\ A(A,\ E_1,\ E_3),\ E_3) = E_2 \Leftrightarrow {}^{\pi_2}A({}^{(12)\pi_2}A,\ E_2,\ E_3) = A(A,\ E_1,\ E_3) \Leftrightarrow {}^{\pi_2}A\ \ \ {}^{132\ \pi_2}A,E_1,E_2\ \ = A\ \ \ {}^{13}\ A,E_1,E_3$, so the identity is equivalent to one of the form III.

55. $A\ \ E_2,\ A\ A,\ E_3,\ E_1\ ,\ \ {}^{123\ \pi_2}A\ \ = E_3 \Leftrightarrow {}^{\pi_2}A\ \ E_2,\ E_3,\ \ {}^{123\ \pi_2}A\ \ = A(A,\ E_3,\ E_1) \Leftrightarrow$ ${}^{(13)\pi_2}A\ \ \ {}^{12\ \pi_2}A, E_1, E_2\ \ = A\ \ \ {}^{13}\ A, E_1, E_3$ – an identity of the form III.

58. $A\ \ A\ E_2,\ E_3,\ A\ ,\ \ {}^{123\ \pi_1}A,\ E_1\ \ = E_2 \Leftrightarrow {}^{\pi_1}A\ \ E_2,\ \ {}^{123\ \pi_1}A,\ E_1\ \ = A\ E_2,\ E_3,\ A\ \ \Leftrightarrow$ ${}^{(12)\pi_1}A\ \ \ {}^{(23)\pi_1}A, E_1, E_2\ \ = A\ \ {}^{(12)}A, E_1, E_3$ – an identity of the form III.

59. $A\ \ A\ E_3,\ E_2,\ A\ ,\ E_2,\ \ {}^{123\ \pi_1}A\ \ = E_1 \Leftrightarrow {}^{\pi_1}A\ \ E_1,\ E_2,\ \ {}^{123\ \pi_1}A\ \ = A\ E_3,\ E_2,\ A\ \ \Leftrightarrow$ ${}^{\pi_1}A\ \ \ {}^{(23)\pi_1}A, E_1, E_2\ \ = A\ \ {}^{(12)}A, E_1, E_3$ – an identity of the form III.

62. $A\ E_2,\ A,\ E_3\ ,\ \ {}^{12\ \pi_1}A,\ E_3\ \ = E_1 \Leftrightarrow {}^{\pi_1}A\ \ \ E_1,\ \ {}^{12\ \pi_1}A,\ E_3\ \ \ = A\ E_2,\ A,\ E_3\ \ \Leftrightarrow$ ${}^{\pi_1}A\ \ \ {}^{(123)\pi_1}A, E_1, E_3\ \ = A\ \ {}^{(13)}A, E_1, E_2$, which is an identity of the form III.

64. $A(A(E_3,\ A,\ E_2),\ E_1,\ {}^{(132)\pi_1}A) = E_3 \Leftrightarrow {}^{\pi_1}A(E_3,\ E_1,\ {}^{(132)\pi_1}A) = A(E_3,\ A,\ E_2) \Leftrightarrow$ ${}^{(123)\pi_1}A\ \ \ {}^{(23)\pi_1}A, E_1, E_3\ \ = {}^{(12)}A\ \ \ {}^{(13)}A, E_1, E_2$ – an identity of the form III.

66. $A\ \ A\ \ A, E_2, E_3\ ,\ E_2,\ E_3\ \ = {}^{\pi_1}A \Leftrightarrow A\ \ {}^{\pi_1}A,\ E_2,\ E_3\ \ = A\ \ A, E_2, E_3\ \ \Leftrightarrow {}^{\pi_1}A\ \ \ {}^{(132)\pi_1}A, E_1, E_2\ \ =$ $A\ \ {}^{(132)}A, E_1, E_2$ – an identity of the form IV.

67. $A\ \ A\ \ A,\ E_3,\ E_2\ ,\ E_2,\ E_3\ \ = {}^{(23)\pi_1}A \Leftrightarrow {}^{\pi_1}A\ \ \ {}^{(23)\pi_1}A,\ E_2,\ E_3\ \ = A\ \ A,\ E_3,\ E_2\ \ \Leftrightarrow$ ${}^{\pi_1}A\ \ \ {}^{(13)\pi_1}A, E_1, E_2\ \ = {}^{(23)}A\ \ \ {}^{(132)}A, E_1, E_2$ – an identity of the form IV.

Taking $A\ \ x_1, x_2, x_3\ \ = y$ in each of the identities 4-17, 19-32, 34, 35, 39, 41, 42, 44, 46, 48-50, 53, 56, 57, 60, 61, 63, 65 and using the parastrophic transformation we obtain, respectively:

4. ${}^{12\ \pi_2}A({}^{\pi_2}A, E_1, E_2) = {}^{(132)\pi_3}A({}^{\pi_3}A, E_1, E_3)$ (an identity of the form III),
5. ${}^{(12)\pi_2}A({}^{\pi_2}A, E_1, E_2) = {}^{(123)\pi_3}A({}^{\pi_3}A, E_1, E_3)$ (the form III),
6. ${}^{123\ \pi_1}A({}^{\pi_1}A, E_1, E_3) = {}^{(13)\pi_2}A({}^{(132)\pi_3}A, E_1, E_2)$ (the form III),
7. ${}^{23\ \pi_2}A({}^{(12)\pi_2}A, E_1, E_2) = {}^{(13)\pi_3}A({}^{\pi_3}A, E_1, E_3)$ (the form III),
8. ${}^{132\ \pi_1}A({}^{\pi_1}A, E_1, E_2) = {}^{(123)\pi_3}A({}^{(12)\pi_3}A, E_1, E_3)$ (the form III),
9. ${}^{12\ \pi_1}A({}^{(132)\pi_1}A, E_1, E_2) = {}^{(23)\pi_2}A({}^{(132)\pi_2}A, E_1, E_2)$ (the form IV),
10. ${}^{12\ \pi_1}A({}^{\pi_1}A, E_1, E_2) = {}^{(132)\pi_3}A({}^{(123)\pi_3}A, E_1, E_3)$ (the form III),
11. ${}^{\pi_3}A({}^{(123)\pi_1}A, E_1, E_2) = {}^{(132)\pi_2}A({}^{(23)\pi_2}A, E_1, E_3)$ (the form III),
12. ${}^{(23)\pi_3}A({}^{(132)\pi_3}A, E_1, E_2) = {}^{(13)\pi_1}A({}^{\pi_1}A, E_1, E_2)$ (the form IV),
13. ${}^{(12)\pi_2}A({}^{(13)\pi_2}A, E_1, E_2) = {}^{(13)\pi_1}A({}^{\pi_1}A, E_1, E_3)$ (the form III),
14. ${}^{(13)\pi_2}A({}^{(132)\pi_2}A, E_1, E_2) = {}^{(123)\pi_1}A({}^{\pi_1}A, E_1, E_3)$ (the form III),
15. ${}^{(132)\pi_1}A({}^{\pi_1}A, E_1, E_2) = {}^{(123)\pi_3}A({}^{(12)\pi_3}A, E_1, E_3)$ (the form III),
16. ${}^{(23)\pi_2}A({}^{(12)\pi_2}A, E_1, E_2) = {}^{(13)\pi_3}A({}^{\pi_3}A, E_1, E_3)$ (the form III),
17. ${}^{\pi_2}A({}^{(123)\pi_2}A, E_1, E_2) = {}^{(132)\pi_1}A({}^{(23)\pi_3}A, E_1, E_3)$ (the form III),
19. ${}^{\pi_1}A({}^{(12)\pi_1}A, E_1, E_2) = {}^{(13)\pi_3}A({}^{(12)\pi_3}A, E_1, E_3)$ (the form III),
20. ${}^{(13)\pi_3}A({}^{(12)\pi_3}A, E_1, E_3) = {}^{\pi_1}A({}^{(12)\pi_1}A, E_1, E_2)$ (the form III),
21. ${}^{(123)\pi_2}A({}^{(12)\pi_2}A, E_1, E_2) = {}^{(132)\pi_3}A({}^{(23)\pi_3}A, E_1, E_2)$ (the form IV),
22. ${}^{(12)\pi_3}A({}^{(123)\pi_3}A, E_1, E_3) = {}^{(23)\pi_1}A({}^{(12)\pi_1}A, E_1, E_2)$ (the form III),
23. ${}^{\pi_2}A({}^{(12)\pi_2}A, E_1, E_2) = {}^{(23)\pi_3}A({}^{(13)\pi_3}A, E_1, E_3)$ (the form III),
24. ${}^{\pi_1}A({}^{(123)\pi_1}A, E_1, E_2) = {}^{(123)\pi_2}A({}^{(12)\pi_2}A, E_1, E_3)$ (the form III),
25. ${}^{(132)\pi_3}A({}^{(123)\pi_3}A, E_1, E_3) = {}^{(12)\pi_1}A({}^{\pi_1}A, E_1, E_2)$ (the form III),
26. ${}^{(13)\pi_1}A({}^{\pi_1}A, E_1, E_3) = {}^{(12)\pi_2}A({}^{(13)\pi_2}A, E_1, E_2)$ (the form III),

27. $^{(132)\pi_2}A(^{\pi_2}A, E_1, E_2) = {}^{\pi_3}A(^{(13)\pi_3}A, E_1, E_3)$ (the form III),
28. $^{(13)\pi_1}A(^{\pi_1}A, E_1, E_2) = {}^{(23)\pi_3}A(^{(132)\pi_3}A, E_1, E_2)$ (the form IV),
29. $^{(13)\pi_2}A(^{\pi_2}A, E_1, E_2) = {}^{(12)\pi_3}A(^{(123)\pi_3}A, E_1, E_2)$ (the form IV),
30. $^{(23)\pi_3}A(^{\pi_3}A, E_1, E_3) = {}^{\pi_2}A(^{(12)\pi_2}A, E_1, E_2)$ (the form III),
31. $^{(23)\pi_1}A(^{(12)\pi_1}A, E_1, E_2) = {}^{(12)\pi_3}A(^{(123)\pi_3}A, E_1, E_3)$ (the form III),
32. $^{(123)\pi_2}A(^{(12)\pi_2}A, E_1, E_3) = {}^{\pi_1}A(^{(123)\pi_1}A, E_1, E_2)$ (the form III),
34. $^{(23)\pi_1}A(^{(13)\pi_1}A, E_1, E_3) = {}^{(132)\pi_2}A(^{(13)\pi_2}A, E_1, E_2)$ (the form III),
35. $^{(132)\pi_2}A(^{(13)\pi_2}A, E_1, E_2) = {}^{(23)\pi_1}A(^{(13)\pi_1}A, E_1, E_3)$ (the form III),
39. $^{(12)\pi_3}A(^{(23)\pi_3}A, E_1, E_2) = {}^{(123)\pi_2}A(^{\pi_2}A, E_1, E_2)$ (the form IV),
41. $^{(23)\pi_3}A(^{(13)\pi_3}A, E_1, E_3) = {}^{\pi_2}A(^{(12)\pi_2}A, E_1, E_2)$ (the form III),
42. $^{(13)}A(A, E_1, E_3) = {}^{\pi_2}A(^{(123)\pi_2}A, E_1, E_2)$ (the form III),
44. $^{(132)\pi_3}A(^{(13)\pi_3}A, E_1, E_2) = {}^{(12)\pi_1}A(^{(23)\pi_1}A, E_1, E_3)$ (the form III),
46. $^{23\ \pi_3}A\ \ ^{13\ \pi_3}A, E_1, E_2 \ = \ ^{123\ \pi_1}A\ \ ^{\pi_1}A, E_1, E_2$ (the form IV),
48. $^{(12)\pi_3}A(^{(23)\pi_3}A, E_1, E_2) = {}^{(123)\pi_2}A(^{\pi_2}A, E_1, E_2)$ (the form IV),
49. $^{(132)}A(^{(23)}A, E_1, E_3) = {}^{(12)\pi_3}A(^{(12)\pi_3}A, E_1, E_2)$ (the form III),
50. $^{\pi_2}A\ \ ^{12\ \pi_2}A, E_1, E_2 \ = \ ^{23\ \pi_3}A\ \ ^{13\ \pi_3}A, E_1, E_3$ (the form III),
53. $^{(12)\pi_1}A\ \ ^{23\ \pi_1}A, E_1, E_2 \ = \ ^{\pi_2}A\ \ ^{132\ \pi_2}A, E_1, E_2$ (the form IV),
56. $^{(13)\pi_2}A\ \ ^{132\ \pi_2}A, E_1, E_2 \ = \ ^{(123)\pi_1}A\ \ ^{\pi_1}A, E_1, E_3$ (the form III),
57. $^{(12)\pi_1}A\ \ ^{\pi_1}A, E_1, E_2 \ = \ ^{(132)\pi_3}A\ \ ^{(123)\pi_3}A, E_1, E_3$ (the form III),
60. $^{(123)\pi_1}A\ \ ^{\pi_1}A, E_1, E_2 \ = \ ^{(23)\pi_3}A\ \ ^{(13)\pi_3}A, E_1, E_2$ (the form III),
61. $A\ \ ^{(12)}A, E_1, E_2 \ = \ ^{(23)\pi_3}A\ \ ^{(132)\pi_3}A, E_1, E_3$ (the form III),
63. $^{(12)\pi_1}A\ \ ^{(23)\pi_1}A, E_1, E_2 \ = \ ^{\pi_2}A\ \ ^{(132)\pi_2}A, E_1, E_2$ (the form IV),
65. $^{(123)\pi_1}A\ \ ^{(23)\pi_1}A, E_1, E_2 \ = \ ^{(13)\pi_2}A\ \ ^{(12)\pi_2}A, E_1, E_3$ (the form III). □

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Note: The work was partially supported by CSSDT ASM grants 15.817.02.26F.

Prezentat la 06.07.2016